

Question 10

Let $f \in C^\infty([-r, r])$, with $r > 0$, and suppose constants $C, A > 0$ satisfy

$$|f^{(n)}(t)| \leq CA^n n! \quad (n \geq 0, |t| \leq r).$$

Write $T_N(x) = \sum_{n=0}^N f^{(n)}(0)x^n/n!$. For a boundary example you may use $g(x) = 1/(1+x^2)$ and the identity

$$g^{(n)}(x) = \frac{(-1)^n n!}{2i} \left((x-i)^{-n-1} - (x+i)^{-n-1} \right), \quad i^2 = -1.$$

Tasks

1. Prove that $T_N(x) \rightarrow f(x)$ for $|x| \leq r$ with $A|x| < 1$. Give a uniform error estimate on $[-\rho, \rho]$ when $0 < \rho \leq r$ and $A\rho < 1$.
2. Explain why this argument gives no conclusion at $A|x| = 1$. Show that both convergence and divergence there are possible under the stated derivative hypothesis, using $f = 1$ and g on $[-1, 1]$ with $C = A = 1$.
3. A function has all derivatives zero at 0 but is positive at every nearby nonzero point. Prove it cannot obey such a derivative bound on any neighborhood of 0.
4. Suppose the displayed bound is known only at $t = 0$. What can you conclude about convergence of the Taylor series, and what additional conclusion can you no longer justify? Supply a counterexample.

Question 10 – Solution

Strategy. A bound on derivatives throughout a segment controls Taylor's remainder; center-only data control coefficients.

Step 1: Force the remainders to vanish. Taylor's theorem uses some ξ between 0 and x . The hypothesis gives

$$|f(x) - T_N(x)| \leq \frac{CA^{N+1}(N+1)!}{(N+1)!}|x|^{N+1} = \boxed{C(A|x|)^{N+1}}.$$

This tends to zero whenever $|x| \leq r$ and $A|x| < 1$. On the specified smaller interval the uniform bound is $C(A\rho)^{N+1} \rightarrow 0$.

Step 2: Test the boundary honestly. At $A|x| = 1$ the bound is only C , which does not tend to zero. The constant function 1 satisfies the bound with $C = A = 1$ and its Taylor series converges at ± 1 . For real x , $|x \pm i| \geq 1$, so the given identity yields $|g^{(n)}(x)| \leq n!$. Yet the Maclaurin series of g is $\sum_{k \geq 0} (-1)^k x^{2k}$; at $x = \pm 1$ its terms fail to tend to zero. Thus neither boundary outcome is forced.

Step 3: Rule out the bound for a flat function. If such constants existed on some $[-r, r]$, choose $0 < |x| < \min(r, 1/A)$. All Taylor polynomials would be zero, but Step 1 would force $f(x) = 0$, contradicting positivity. Thus no finite positive C, A can work on any such neighborhood.

Step 4: Separate coefficients from function values. A bound only at 0 gives $|f^{(n)}(0)/n!| \leq CA^n$. Geometric comparison proves absolute convergence of the Taylor series for $|x| < 1/A$, but does not prove that its sum is $f(x)$. The smooth function e^{-1/x^2} extended by zero at 0 has every center derivative zero, so it satisfies every such center-only bound; its Taylor series is zero while the function is positive elsewhere.