

Question 9

Let $T_4(x) = 1 - x^2/2 + x^4/24$, the fourth-degree Maclaurin polynomial of $\cos x$. Consider a rational competitor

$$Q(x) = \frac{1 + ax^2}{1 + bx^2}, \quad a, b \in \mathbb{R}.$$

A student believes that replacing a Taylor polynomial by a rational expression with the same local coefficients must improve accuracy.

Tasks

1. Determine a, b so that Q and $\cos x$ agree through degree four at 0, and identify any real poles of Q .
2. Find the leading nonzero terms of $\cos x - T_4(x)$ and $\cos x - Q(x)$. Compute the limiting ratio of their absolute errors as $x \rightarrow 0, x \neq 0$.
3. Use an alternating Taylor enclosure for $\cos 1$ to prove which approximation is closer at $x = 1$, without relying on a calculator value of $\cos 1$.
4. Evaluate the student's claim and explain why matching coefficients alone cannot rank all approximations on a whole interval.

Question 9 – Solution

Strategy. Match coefficients, then examine the first coefficient on which the competing approximations disagree.

Step 1: Solve the matching equations. Near 0, $Q = 1 + (a-b)x^2 + (b^2-ab)x^4 + \dots$. Thus $a-b = -1/2$ and $-b(a-b) = 1/24$, giving

$$b = \frac{1}{12}, \quad a = -\frac{5}{12}, \quad Q(x) = \frac{12-5x^2}{12+x^2}.$$

Since $12+x^2 > 0$, there are no real poles.

Step 2: Compare the leading errors. The rational geometric expansion has x^6 coefficient $-1/288$. Hence

$$\cos x - T_4(x) = -\frac{x^6}{720} + O(x^8), \quad \cos x - Q(x) = \frac{x^6}{480} + O(x^8).$$

Therefore $\lim_{x \rightarrow 0} \frac{|\cos x - Q(x)|}{|\cos x - T_4(x)|} = 3/2$. In particular, the rational approximation is worse for all sufficiently small nonzero x .

Step 3: Make an exact comparison at one. Alternating bounds give

$$T_6(1) = \frac{389}{720} < \cos 1 < T_4(1) = \frac{13}{24}, \quad Q(1) = \frac{7}{13}.$$

Here $Q(1) < 389/720$, and the midpoint of the two approximations is $\frac{1}{2}(7/13 + 13/24) = 337/624 < 389/720$. Thus $\cos 1$ lies above their midpoint and between them: $T_4(1)$ is strictly closer than $Q(1)$.

Step 4: State the proper conclusion. The claim fails both locally and at $x = 1$. Matching four derivatives fixes contact at the center, not the next error coefficient or behavior far away. For $0 < |x| \leq 1$, alternating bounds give $\cos x - T_4(x) < 0$ and

$$\cos x - Q(x) \geq x^6 \left(\frac{1}{312} - \frac{1}{720} \right) > 0,$$

since $T_4 - Q = x^6/[288(1+x^2/12)]$. These signs explain the plotted errors.

