

Question 8

An unknown function belongs to the family $f(x) = e^{ax} \cos(bx)$, where a, b are real. Exact local measurements give

$$f'(0) = 1, \quad f''(0) = -3.$$

A separate report claims $f'''(0) = -10$. No measurement uncertainty is assumed.

Tasks

1. Recover every admissible parameter pair (a, b) from the first two measurements. Explain the remaining ambiguity and whether it changes f .
2. Find the fourth-degree Maclaurin polynomial of every compatible function.
3. Test the separate third-derivative report and decide whether all three measurements can arise from one member of the stated family.
4. Prove a uniform remainder bound below 5.2×10^{-6} for your polynomial on $[-0.1, 0.1]$. You may use $f(x) = \operatorname{Re} e^{(a+ib)x}$ and $|\operatorname{Re} z| \leq |z|$.

Question 8 – Solution

Strategy. Turn measured derivatives into algebraic constraints before predicting higher coefficients.

Step 1: Recover the parameters. Differentiation at 0 gives $f'(0) = a$ and $f''(0) = a^2 - b^2$. Hence $a = 1$ and $b^2 = 4$, so

$$(a, b) = (1, 2) \text{ or } (1, -2).$$

The sign cannot be recovered because cosine is even; both pairs produce the same function $e^x \cos 2x$.

Step 2: Predict the Taylor polynomial. Using real parts of $(1 + 2i)^n$, the derivative values for $n = 0, \dots, 4$ are $1, 1, -3, -11, -7$. Dividing by $n!$ yields

$$T_4(x) = 1 + x - \frac{3}{2}x^2 - \frac{11}{6}x^3 - \frac{7}{24}x^4.$$

For example, $(1 + 2i)^2 = -3 + 4i$ and $(1 + 2i)^3 = -11 - 2i$.

Step 3: Reject inconsistent exact data. Every pair compatible with the first two measurements forces $f'''(0) = -11$. The reported value -10 is incompatible. Changing the sign of b conjugates the complex powers and leaves their real parts unchanged, so it cannot repair the discrepancy.

Step 4: Bound the fifth derivative uniformly. For real t ,

$$|f^{(5)}(t)| = \left| \operatorname{Re} \left((1 + 2i)^5 e^{(1+2i)t} \right) \right| \leq 5^{5/2} e^t.$$

Taylor's theorem on each segment from 0 to $x \in [-0.1, 0.1]$ therefore gives

$$|f(x) - T_4(x)| \leq \frac{5^{5/2} e^{0.1}}{120} 10^{-5} < 5.2 \times 10^{-6}.$$

For a purely rational check of the last inequality, use $\sqrt{5} < 9/4$ and $e^{0.1} < 1.106$: the latter follows by bounding the tail after $1 + 0.1 + 0.1^2/2$ by $(0.1^3/6)/(1 - 0.025)$. These bounds give less than 5.185×10^{-6} .