

Question 7

The direct Maclaurin expansion of $\ln(1+x)$ converges slowly at $x=1$. Seek a faster Taylor-series evaluation of $\ln 2$ using

$$L(z) = \ln\left(\frac{1+z}{1-z}\right), \quad |z| < 1.$$

Count terms by the number of nonzero summands, not by polynomial degree.

Tasks

1. Derive the Maclaurin series for L by differentiating and integrating a geometric series, with a justification on $|z| < 1$.
2. Select a positive z for which $L(z) = \ln 2$. For the first N terms, prove the tail bound

$$\frac{2z^{2N+1}}{(2N+1)(1-z^2)}$$

and determine the error sign.

3. Find the smallest N for which this bound is strictly below 10^{-6} at the selected z . Give the resulting approximation and a rigorous enclosure.
4. For the direct alternating harmonic sum with N terms, how many terms does the next-term bound require to certify error strictly below 10^{-6} ? Compare the two certificates and explain the limits of this comparison.

Question 7 – Solution

Strategy. Change the evaluation point to shrink the geometric factor controlling the Taylor tail.

Step 1: Construct the odd-power series. Since $L'(z) = 2/(1 - z^2) = 2 \sum_{k \geq 0} z^{2k}$ and $L(0) = 0$,

$$L(z) = 2 \sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1}, \quad |z| < 1.$$

The geometric series converges uniformly on any closed segment inside $(-1, 1)$, so termwise integration is justified.

Step 2: Bound a positive tail. The equation $(1+z)/(1-z) = 2$ gives $z = 1/3$. With $A_N = 2 \sum_{k=0}^{N-1} z^{2k+1}/(2k+1)$ and $0 < z < 1$,

$$\begin{aligned} 0 < L(z) - A_N &= 2 \sum_{k=N}^{\infty} \frac{z^{2k+1}}{2k+1} \\ &< \frac{2z^{2N+1}}{(2N+1)(1-z^2)} =: B_N. \end{aligned}$$

Strictness follows because later denominators exceed $2N+1$.

Step 3: Count the accelerated terms. At $z = 1/3$, $B_N = 9/[4(2N+1)3^{2N+1}]$, strictly decreasing in N . Here $B_5 = 9/7794468 > 10^{-6}$ and $B_6 = 9/82904796 < 1.086 \times 10^{-7}$. Thus six terms are the smallest number certified by this bound, and

$$A_6 < \ln 2 < A_6 + \frac{9}{82904796}, \quad A_6 \approx 0.6931470738.$$

Step 4: Make a fair certificate comparison. For the direct alternating sum with N terms, the stated bound is $1/(N+1)$. To make that bound strictly less than 10^{-6} requires $N \geq 1,000,000$. The comparison is six versus one million terms under these particular certificates. It does not assert that a million terms are necessary for the true direct-series error, nor that a loose bound is an optimal algorithm. The transformed series profits from the factor $z^2 = 1/9$.