

Question 6

Define a function on the real line by

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

A student claims: “Every infinitely differentiable function agrees with its Taylor series near the center.”

Tasks

1. Prove $|x|^{-m}e^{-1/x^2} \rightarrow 0$ as $x \rightarrow 0$ for every nonnegative integer m .
2. Prove that f is infinitely differentiable at 0 and that every derivative there vanishes. Use a polynomial-in- $1/x$ description away from 0 and difference quotients at 0.
3. Find the Maclaurin series, its radius of convergence, and the set of points where it equals f .
4. Explain precisely why Taylor’s theorem is still valid but fails to prove equality to the infinite series here. Identify the remainder $R_n(x)$ at any fixed nonzero x .

Question 6 – Solution

Strategy. Separate the existence of every derivative from the limiting behavior of the Taylor remainders.

Step 1: Exponential decay beats every power. Put $u = 1/x^2 \rightarrow \infty$. Choose an integer $k > m/2$. Since $e^u \geq u^k/k!$,

$$|x|^{-m} e^{-1/x^2} = u^{m/2} e^{-u} \leq k! u^{m/2-k} \rightarrow 0.$$

Step 2: Prove smoothness, including the center. For $x \neq 0$, induction gives $f^{(n)}(x) = P_n(1/x)e^{-1/x^2}$, where $P_0(z) = 1$ and

$$P_{n+1}(z) = -z^2 P'_n(z) + 2z^3 P_n(z).$$

Define each candidate derivative to be zero at 0. Step 1 proves its continuity there. Its difference quotient at 0 is $P_n(1/h)e^{-1/h^2}/h \rightarrow 0$, again by Step 1. Inductively each candidate is the derivative of the preceding one, so $f \in C^\infty(\mathbb{R})$ and $f^{(n)}(0) = 0$ for every $n \geq 0$.

Step 3: Compare the series with the function. Every Taylor polynomial is zero. Thus the Maclaurin series is $\sum_{n=0}^{\infty} 0x^n = 0$, with infinite radius. It equals f only at $x = 0$, since $f(x) > 0$ for every nonzero x . Infinite radius does not by itself identify the represented function.

Step 4: Locate the missing hypothesis. Taylor's finite theorem remains valid on the segment from 0 to any fixed x . But $R_n(x) = f(x) - T_n(x) = e^{-1/x^2} > 0$ for $x \neq 0$, independently of n . Thus $R_n(x) \not\rightarrow 0$. Smoothness provides each finite remainder formula, not a derivative-growth bound forcing remainders to vanish. The graph and the zero Taylor sum touch to every order at 0 without agreeing nearby.

