

Question 5

Consider the nonelementary integral

$$I = \int_0^{1/2} e^{-t^2} dt, \quad S_m = \sum_{k=0}^m \frac{(-1)^k}{2^{2k+1} k! (2k+1)}.$$

An accuracy certificate is required; a decimal from a calculator is insufficient.

Tasks

1. Derive the series for I and justify integrating the Maclaurin series term by term on the whole integration interval.
2. Find the smallest m for which the first-omitted-term bound certifies $|I - S_m| \leq 10^{-6}$, and state the number of terms used.
3. Give an exact rational enclosure for I using that partial sum and the next one. Report an approximation with its certified absolute error.
4. Explain the sign of the error geometrically by comparing e^{-t^2} with the degree- $2m$ polynomial under the integral on $[0, 1/2]$.

Question 5 – Solution

Strategy. Integrate a uniformly convergent series, then use the integrated alternating tail as an error certificate.

Step 1: Justify the interchange. On $[0, 1/2]$, $|(-t^2)^k/k!| \leq (1/4)^k/k!$, whose sum is finite. The Weierstrass test gives uniform absolute convergence. Integrating yields

$$I = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+1}k!(2k+1)}.$$

The positive term magnitudes decrease strictly to zero.

Step 2: Choose a certified truncation. The first omitted magnitude is $b_m = 1/[2^{2m+3}(m+1)!(2m+3)]$. It decreases with m . For $m = 3$, $b_3 = 1/110592 > 10^{-6}$, whereas

$$b_4 = \frac{1}{2703360} < 3.70 \times 10^{-7} < 10^{-6}.$$

Thus $\boxed{m = 4}$ is the smallest index certified by this bound, using five terms.

Step 3: Supply an exact enclosure. The even partial sum is an upper bound and the next odd sum a lower bound:

$$S_4 - \frac{1}{2703360} = S_5 < I < S_4 = \frac{1}{2} - \frac{1}{24} + \frac{1}{320} - \frac{1}{5376} + \frac{1}{110592}.$$

Numerically, $S_4 \approx 0.4612813637$. Reporting $I \approx 0.46128136$ introduces less than 4×10^{-9} additional rounding error, so its total absolute error is less than 3.74×10^{-7} .

Step 4: Interpret the signed area. Set $p_8(t) = 1 - t^2 + t^4/2 - t^6/6 + t^8/24$. The alternating tail is strictly negative for $t > 0$ in this interval, so $p_8(t) > e^{-t^2}$. Hence $S_4 - I$ is the positive area between the curves. The scaled gap below makes the very small difference visible; it vanishes at $t = 0$.

