

Question 4

Let $F(x) = e^{x^2} \cos x$. A product of truncated series contains both useful terms and terms beyond the intended degree.

Tasks

1. Obtain the sixth-degree Maclaurin polynomial P_6 of F by explicit coefficient convolution.
2. Find $F^{(6)}(0)$ and explain why the remainder has order x^8 , rather than merely order x^7 .
3. Write $A = 1 + x^2 + x^4/2 + x^6/6$ and $B = 1 - x^2/2 + x^4/24 - x^6/720$. Compute $AB - P_6$ exactly.
4. On $|x| \leq 1/2$, prove the concrete bound $|F(x) - P_6(x)| < 0.0005$. Account separately for the exponential tail, cosine tail, and high-degree terms in AB .

Question 4 – Solution

Strategy. Convolve by degree, then bound the two truncation errors and the discarded product terms separately.

Step 1: Collect only the requested powers. Using the entire exponential and cosine series,

$$[x^2]F = 1 - \frac{1}{2} = \frac{1}{2}, \quad [x^4]F = \frac{1}{2} - \frac{1}{2} + \frac{1}{24} = \frac{1}{24},$$

$$[x^6]F = \frac{1}{6} - \frac{1}{4} + \frac{1}{24} - \frac{1}{720} = -\frac{31}{720}.$$

Thus $\boxed{P_6(x) = 1 + x^2/2 + x^4/24 - 31x^6/720}$.

Step 2: Recover a derivative and its parity. We have $F^{(6)}(0) = 6!(-31/720) = -31$. Both factors are even and analytic everywhere, so all odd coefficients vanish. The next possible power after degree six is eight, proving $F - P_6 = O(x^8)$ near 0.

Step 3: Retain the discarded product explicitly. Direct multiplication gives

$$AB - P_6 = -\frac{23}{360}x^8 + \frac{1}{160}x^{10} - \frac{1}{4320}x^{12}.$$

These terms belong to the product of polynomials, not to P_6 .

Step 4: Certify the error on the whole interval. Write $E = e^{x^2}$ and $C = \cos x$. Then $EC - P_6 = (E - A)C + A(C - B) + (AB - P_6)$. For $|x| \leq 1/2$, $|C| \leq 1$, $0 < A \leq e^{1/4} < 4/3$ (compare the exponential series with the geometric series). Taylor's theorem and the alternating cosine tail give

$$|E - A| \leq \frac{4}{3} \frac{|x|^8}{24}, \quad |C - B| \leq \frac{|x|^8}{8!}.$$

Consequently,

$$|F - P_6| \leq \frac{4/3}{24 \cdot 256} + \frac{4/3}{40320 \cdot 256} + \frac{23}{360 \cdot 256} + \frac{1}{160 \cdot 1024} + \frac{1}{4320 \cdot 4096} < 0.000473 < 0.0005.$$

The bound is conservative but rigorous; no unbounded O -term is used as a numerical certificate.