

Question 3

For $x \neq 0$, define

$$H(x) = \frac{\sin x - x \cos x}{x^3}.$$

Near the origin the numerator suffers substantial cancellation.

Tasks

1. Derive the full Maclaurin series of H by combining the sine and cosine series before dividing. State its convergence domain.
2. Determine the unique continuous extension at 0, and find $H''(0)$ and $H^{(4)}(0)$ for that extension.
3. Construct its fourth-degree Taylor polynomial and certify its absolute error on $[-1, 1]$ by an alternating-series estimate.
4. Prove that the extended function is even and strictly positive on $[-1, 1]$. Explain why the series is useful for evaluating the original quotient near 0.

Question 3 – Solution

Strategy. Expose the first nonzero numerator term before removing the apparent singularity.

Step 1: Cancel and reindex. The coefficient of x^{2k+1} in the numerator is $(-1)^k[1/(2k+1)! - 1/(2k)!]$. It is zero when $k = 0$. Setting $j = k - 1$ for the surviving terms gives

$$H(x) = \sum_{j=0}^{\infty} \frac{(-1)^j 2(j+1)}{(2j+3)!} x^{2j}.$$

Sine and cosine converge absolutely for every real x ; the new series also has infinite radius by the ratio test. It agrees with the quotient when $x \neq 0$.

Step 2: Fill the removable singularity. The new series defines an analytic extension with

$$H(0) = \frac{1}{3}, \quad H''(0) = 2! \left(-\frac{1}{30} \right) = -\frac{1}{15}, \quad H^{(4)}(0) = 4! \frac{1}{840} = \frac{1}{35}.$$

Continuity forces the value $1/3$, so no other continuous extension is possible.

Step 3: Bound the truncated tail. The polynomial is $T_4(x) = 1/3 - x^2/30 + x^4/840$. For $|x| \leq 1$, successive term magnitudes have ratio $[(j+2)/(j+1)]x^2/[(2j+5)(2j+4)] \leq 1/10$. They decrease to zero, so

$$|H(x) - T_4(x)| \leq \frac{|x|^6}{45360} \leq \frac{1}{45360}.$$

Step 4: Interpret symmetry and stability. Only even powers occur. Alternating bounds give $H(x) \geq 1/3 - x^2/30 \geq 3/10 > 0$ on $[-1, 1]$. The numerator subtracts two quantities of order x to obtain one of order x^3 ; the series evaluates the finite limiting value without that subtraction. This is a cancellation concern, not a failure of the exact quotient.

