

Question 2

For $0 \leq x \leq 1$, let $T_n(x) = \sum_{k=0}^n (-x)^k / k!$ approximate e^{-x} . The number of terms in T_n is $n + 1$.

Tasks

1. Prove that the Maclaurin series represents e^{-x} on $[0, 1]$, using a remainder estimate that is uniform in x .
2. Find the smallest degree certified by the bound $|e^{-x} - T_n(x)| \leq 1/(n + 1)!$ to achieve error at most 10^{-6} throughout $[0, 1]$.
3. A sufficient bound need not identify the true smallest degree. Prove in this case that the degree you found really is minimal, using the alternating tail at $x = 1$ to exclude every smaller degree.
4. Produce a rational enclosure for e^{-1} from consecutive partial sums at that degree, and explain which endpoint is the upper bound.

Question 2 – Solution

Strategy. Use Taylor's theorem for convergence and alternating tails for both upper and lower error bounds.

Step 1: Control all points simultaneously. Every derivative of e^{-x} has absolute value $e^{-x} \leq 1$ on $[0, 1]$. Taylor's theorem gives

$$|e^{-x} - T_n(x)| \leq \frac{x^{n+1}}{(n+1)!} \leq \frac{1}{(n+1)!} \rightarrow 0.$$

This proves equality to the series and uniform convergence on the stated interval.

Step 2: Find a certified degree. Because $9! = 362880 < 10^6 < 3628800 = 10!$, the smallest degree certified by this bound is $\boxed{n=9}$, using ten terms. Degree 8 is not certified by the upper bound alone.

Step 3: Prove actual minimality. At $x = 1$, the alternating tail has the sign of its first term, and its magnitude is strictly greater than the difference of its first two magnitudes:

$$|e^{-1} - T_n(1)| > \frac{1}{(n+1)!} - \frac{1}{(n+2)!} = \frac{n+1}{(n+2)!} =: b_n.$$

Indeed, the remaining tail starts positive after factoring out its first sign and grouping consecutive pairs. Also $b_{n+1}/b_n = (n+2)/[(n+1)(n+3)] < 1$ for $n \geq 0$. Hence for every $0 \leq n \leq 8$, the error exceeds $b_8 = 9/10! > 10^{-6}$. No smaller degree works on all of $[0, 1]$.

Step 4: Enclose the value rationally. Odd partial sums lie below the limit and even ones above. Therefore

$$\boxed{\frac{16687}{45360} = T_9(1) < e^{-1} < T_{10}(1) = \frac{16481}{44800}}.$$

The interval width is exactly $1/10!$; the upper endpoint comes from the positive first omitted term after T_9 .