

Question 1

Let $f(x) = \ln x$, and expand about $x = 2$. A student proposes

$$P_4(x) = \ln 2 + \sum_{n=1}^4 \frac{(-1)^{n-1}(n-1)!}{2^n} (x-2)^n.$$

A Taylor coefficient and a derivative value are different kinds of data.

Tasks

1. Derive $f^{(n)}(2)$ for $n \geq 1$, correct the proposed polynomial, and identify its first incorrect coefficient.
2. Find the full Taylor series and its interval of convergence, including both endpoints. Justify that its sum equals $\ln x$ on the open interval.
3. Use the Lagrange remainder to give a uniform absolute error bound for the corrected fourth-degree polynomial on $[1.8, 2.2]$.
4. Decide whether the corrected polynomial lies above or below $\ln x$ on each side of 2 in that interval. Support the conclusion with a signed remainder.

Question 1 – Solution

Strategy. Divide derivatives by factorials, then distinguish series convergence from finite-polynomial accuracy.

Step 1: Normalize the coefficients. For $n \geq 1$, $f^{(n)}(x) = (-1)^{n-1}(n-1)!/x^n$. Thus

$$T_4(2+h) = \ln 2 + \frac{h}{2} - \frac{h^2}{8} + \frac{h^3}{24} - \frac{h^4}{64}.$$

The student omitted division by $n!$; the first error is the h^2 coefficient, $-1/4$ instead of $-1/8$.

Step 2: Establish the represented function. Integrating the geometric series for $1/(2+h)$ from 0 to h gives $\ln(2+h) = \ln 2 + \sum_{n \geq 1} (-1)^{n-1} h^n / (n 2^n)$ for $|h| < 2$. Uniform convergence on the integration segment justifies this step. The term ratio tends to $|h|/2$, so the radius is 2; $h = -2$ gives the divergent negative harmonic series and $h = 2$ gives the alternating harmonic series. The convergence interval is $(0, 4]$ in x .

Step 3: Certify a uniform error. Since $f^{(5)}(x) = 24/x^5$, on $[1.8, 2.2]$,

$$|f(x) - T_4(x)| \leq \frac{24}{1.8^5} \frac{0.2^5}{120} = \frac{1}{5 \cdot 9^5} < 3.39 \times 10^{-6}.$$

Step 4: Determine the error sign. For $x \neq 2$, $f(x) - T_4(x) = h^5/(5\xi^5)$ for some positive ξ between 2 and x . Thus T_4 is above f on $[1.8, 2)$ and below it on $(2, 2.2]$; equality holds at 2. The plot magnifies this signed error.

