

Question 10

An unknown real power series $\sum_{n=0}^{\infty} a_n(x-2)^n$ converges at $x = 5$ and diverges at $x = -2$. No coefficient formula is supplied. Use only what these two observations imply; boundary convergence may be conditional.

Tasks

1. Find the strongest possible bounds on its radius of convergence R . Explain why either inequality may be an equality.
2. Classify the behavior forced at $x = 0, -1, 11/2, 6, 7$: absolute convergence, divergence, or not determined by the information. Give reasons using distances from the center.
3. Construct two explicit series satisfying both observations that establish all the undetermined cases in your table. Verify the observations and the disputed points for each example.
4. Show every radius in your proposed closed range can occur. Use explicit examples for the endpoints and a general construction for the intermediate radii.

Question 10 – Solution

Strategy. Convergence at a point controls smaller distances absolutely; divergence controls larger distances. Boundary tests cannot be transferred automatically to the opposite side.

Step 1: Bound the radius sharply. The convergent point is distance 3 from the center, so $R \geq 3$. The divergent point is distance 4, so $R \leq 4$: if $R > 4$ that point would be inside the region of absolute convergence. Thus $\boxed{3 \leq R \leq 4}$. Convergence at distance 3 could be boundary convergence, and divergence at distance 4 could be boundary divergence.

Step 2: Record only forced conclusions.

x	$ x - 2 $	conclusion
0	2	absolute convergence
-1	3	not determined
11/2	7/2	not determined
6	4	not determined
7	5	divergence

The absolute convergence at distance 2 follows from the convergent point at distance 3. The opposite point at distance 3 need not share an endpoint test, and distances between 3 and 4 are not settled by the bounds.

Step 3: Construct examples proving the uncertainty. Take, starting at $n = 1$ and setting $a_0 = 0$,

$$U(x) = \sum_{n \geq 1} \frac{(-1)^n (x - 2)^n}{n 3^n}, \quad V(x) = \sum_{n \geq 1} \frac{(-1)^n (x - 2)^n}{n 4^n}.$$

For U , $x = 5$ gives alternating harmonic convergence and $x = -2$ fails the term test; $R_U = 3$. At $x = -1$ it is positive harmonic and diverges, and at 11/2 and 6 its terms do not tend to zero. For V , $x = 5$ converges absolutely, while $x = -2$ is positive harmonic and diverges; $R_V = 4$. At -1 and 11/2 it converges absolutely, and at 6 it converges conditionally. These two examples resolve every “not determined” entry without changing the given observations.

Step 4: Realize the entire radius range. The examples above realize $R = 3$ and $R = 4$. For any

$3 < \rho < 4$, use $\boxed{\sum_{n=0}^{\infty} ((x - 2)/\rho)^n}$. Its radius is ρ ; at 5 its ratio has magnitude $3/\rho < 1$, and at -2 it has magnitude $4/\rho > 1$. Thus every radius in $[3, 4]$ occurs, and no narrower range follows from the two observations alone.