

Question 8

Consider a series of polynomials,

$$P(x) = \sum_{n=0}^{\infty} \left(\frac{x^2 - 4}{2} \right)^n.$$

It is geometric in $(x^2 - 4)/2$, but its displayed terms are not single monomials in x . Investigate how its convergence set compares with that of an ordinary power series representing the same sum.

Tasks

1. Determine the exact real convergence set of the displayed series, including all boundary tests. Explain why it need not be one interval centered at zero.
2. Sum the series wherever it converges. Compare the domain of that rational formula with the convergence set just found.
3. Starting afresh from the rational formula, derive an ordinary power series centered at zero. Find its radius and endpoint behavior.
4. Compare the two series at $x = 0$ and $x = 1$, and state exactly where their sums are known to agree. Diagram both convergence sets and explain why one should not silently substitute the new domain for the original one.

Question 8 – Solution

Strategy. Apply the geometric test to the substituted variable first; then derive and test a separate series in powers of x .

Step 1: Pull back the geometric convergence condition. The displayed series converges exactly when $|(x^2 - 4)/2| < 1$, or $2 < x^2 < 6$. Thus its convergence set is

$$(-\sqrt{6}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{6}).$$

At $x = \pm\sqrt{2}$ the ratio is -1 ; at $x = \pm\sqrt{6}$ it is 1 . In each case the terms fail to approach zero. Elsewhere outside the two intervals the ratio has magnitude greater than one. These polynomial terms do not have the ordinary form $a_n(x - c)^n$ for a fixed center.

Step 2: Sum on the justified set. Geometric summation gives $P(x) = 2/(6 - x^2)$ on the two intervals above. This rational formula is defined for every real $x \neq \pm\sqrt{6}$, which is a much larger set. Its being finite at other points says nothing by itself about convergence of the displayed polynomial series there.

Step 3: Construct a distinct centered power series. Factoring the denominator differently gives

$$\frac{2}{6 - x^2} = \frac{1/3}{1 - x^2/6} = \sum_{n=0}^{\infty} \frac{x^{2n}}{3 \cdot 6^n}, \quad |x| < \sqrt{6}.$$

Its odd coefficients vanish. Its radius is $\sqrt{6}$ and both endpoints diverge because the terms there equal $1/3$. The new convergence interval is $(-\sqrt{6}, \sqrt{6})$.

Step 4: Compare actual sums, not just formulas. At $x = 0$, the original terms are $(-2)^n$, so the original series diverges; the new series sums to $1/3$. At $x = 1$, the original ratio is $-3/2$ and its series diverges, while the new series sums to $2/5$. On $\sqrt{2} < |x| < \sqrt{6}$, both converge and both equal $2/(6 - x^2)$. The new representation adds valid points; it does not retroactively change the convergence set of the original sequence of partial sums.

