

Question 7

Suppose $y(x) = \sum_{n=0}^{\infty} a_n(x-2)^n$ has positive radius R . Consider $H(x) = xy'(x) + y(x)$ only for $|x-2| < R$. A student claims that $H(x) = \sum_{n=0}^{\infty} (n+1)a_n(x-2)^n$.

Tasks

1. Derive the correct coefficient of $(x-2)^n$ in H for every $n \geq 0$, reindexing the derivative carefully.
2. Write the constant, linear, quadratic and cubic coefficients explicitly in terms of the a_n .
3. Identify the student's missing contribution and characterize exactly which series y make the claimed identity valid throughout the open interval.
4. Apply your formula to $a_n = 3^{-n-1}$. Sum y geometrically, compute H directly from this closed form, and verify the coefficients agree. State the resulting series' radius.

Question 7 – Solution

Strategy. Work in $h = x - 2$ and remember that multiplication by x means multiplication by $h + 2$, not merely by h .

Step 1: Reindex before multiplying. Inside the radius, $y' = \sum_{n=0}^{\infty} (n+1)a_{n+1}h^n$. Now $hy' = \sum_{n=1}^{\infty} na_nh^n$; its constant coefficient is zero. Adding $2y'$ and y yields

$$H = \sum_{n=0}^{\infty} ((n+1)a_n + 2(n+1)a_{n+1})h^n.$$

The formula also works for $n = 0$, where the constant term of xy' comes entirely from $2y'$.

Step 2: Display the first four coefficients. The requested coefficients are

$$a_0 + 2a_1, \quad 2a_1 + 4a_2, \quad 3a_2 + 6a_3, \quad 4a_3 + 8a_4.$$

As a direct constant-term check, $H(2) = 2y'(2) + y(2) = 2a_1 + a_0$. Keeping the same summation index through a differentiation would miss the shift from a_n to a_{n+1} .

Step 3: Determine exactly when the shortcut works. The claimed series actually represents $hy' + y$. Its difference from H is $2y'$. The identity therefore holds throughout the interval exactly when $y' \equiv 0$, or equivalently $a_n = 0$ for every $n \geq 1$. One justification of the equivalence is uniqueness of power-series coefficients, obtained by successive differentiation at the center. Any constant a_0 is allowed. Equality at a single isolated point would not imply this identity.

Step 4: Check against a geometric closed form. For $a_n = 3^{-n-1}$, $y = (1/3) \sum (h/3)^n = 1/(3-h) = 1/(5-x)$ when $|h| < 3$. Hence

$$H = \frac{x}{(5-x)^2} + \frac{1}{5-x} = \frac{5}{(5-x)^2}.$$

The coefficient formula gives $(n+1)/3^{n+1} + 2(n+1)/3^{n+2} = 5(n+1)/3^{n+2}$. This agrees with the differentiated geometric identity $5/(3-h)^2 = \sum_{n \geq 0} 5(n+1)h^n/3^{n+2}$. The radius is $\boxed{3}$; the factors growing linearly in n do not change it.