

Question 6

Let $f(x) = 1/(4 - x)$, defined for real $x \neq 4$. Represent this same function by geometric power series with two different centers, $c_1 = 1$ and $c_2 = -1$. Do not use a Taylor-coefficient formula.

Tasks

1. Rewrite the denominator about each center and obtain the two power series explicitly.
2. Find both radii and exact real convergence intervals, testing their endpoints. Mark centers and intervals in a diagram.
3. At $x = -4$, decide which series converges and find its sum. At $x = 5$, decide whether either series converges even though $f(5)$ exists.
4. Explain why the two series agree wherever both converge, why changing the center changes the radius here, and why the domain of the rational function is not the convergence interval of either series.

Question 6 – Solution

Strategy. Factor out the distance from each center to the denominator's zero, then apply the geometric-series condition to the normalized displacement.

Step 1: Expand about each center. Using $4 - x = 3 - (x - 1)$ and $4 - x = 5 - (x + 1)$ gives

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-1)^n}{3^{n+1}}, \quad f(x) = \sum_{n=0}^{\infty} \frac{(x+1)^n}{5^{n+1}}.$$

Each equality is initially asserted only when its geometric ratio has magnitude less than one.

Step 2: Determine the two intervals. The first ratio is $(x-1)/3$, so $R_1 = 3$ and $I_1 = (-2, 4)$. The second is $(x+1)/5$, so $R_2 = 5$ and $I_2 = (-6, 4)$. At each endpoint the terms have constant nonzero magnitude ($1/3$ or $1/5$), so all four endpoint tests give divergence. Both series converge absolutely inside their respective intervals.

Step 3: Test the specified points. At $x = -4$, the first ratio is $-5/3$, so its terms do not tend to zero. The second ratio is $-3/5$, so it converges and sums to $\boxed{(1/5)/(1+3/5) = 1/8 = f(-4)}$. At $x = 5$, the ratios are $4/3$ and $6/5$, so both series diverge even though $\boxed{f(5) = -1}$ is a valid rational-function value.

Step 4: Compare representations and domains. On the common interval $(-2, 4)$, each convergent geometric identity gives the same function $1/(4-x)$, so their sums agree. Their right endpoints coincide with the denominator's zero, but this point is distance 3 from one center and distance 5 from the other. The left boundaries follow from the same absolute-ratio conditions, even though f is finite there. The rational function has domain $\mathbb{R} \setminus \{4\}$, while each series represents it on only its own convergence interval. Evaluation of the rational formula outside that interval is not evaluation of the series.

