

Question 5

Consider two series centered at zero:

$$A(x) = \sum_{n=0}^{\infty} x^n, \quad B(x) = 1 - \sum_{n=1}^{\infty} \frac{x^n}{2^n}.$$

Let $C(x) = \sum_{n=0}^{\infty} c_n x^n$ be the series obtained by their Cauchy product, using finite coefficient convolutions.

Tasks

1. Find the sums and exact convergence intervals of A and B separately.
2. Compute c_n for every $n \geq 0$, including the constant coefficient. Determine the radius and interval of the resulting series C .
3. Identify the interval where the Cauchy-product theorem directly proves $C = AB$. Explain the rational cancellation that permits the new series to converge on a larger interval.
4. Evaluate the series C at $x = 1$ and $x = 3/2$. At each point, decide whether this value can legitimately be called the product of the two original series sums.

Question 5 – Solution

Strategy. Compute the product coefficients before interpreting the enlarged domain; a new convergent series does not make a divergent factor converge.

Step 1: Sum and test the two factors. Geometric summation gives

$$A(x) = \frac{1}{1-x} \quad (|x| < 1), \quad B(x) = 1 - \frac{x/2}{1-x/2} = \frac{1-x}{1-x/2} \quad (|x| < 2).$$

At each respective endpoint the terms fail to tend to zero. Thus the exact intervals are $(-1, 1)$ and $(-2, 2)$, both absolutely convergent inside.

Step 2: Compute the finite coefficient sums. The coefficients of A are all 1; those of B are $b_0 = 1$ and $b_n = -2^{-n}$ for $n \geq 1$. Hence

$$c_n = \sum_{k=0}^n b_k = 1 - \sum_{k=1}^n 2^{-k} = 2^{-n} \quad (n \geq 0).$$

In particular $c_0 = 1$. Therefore $C(x) = \sum_{n \geq 0} (x/2)^n = 1/(1-x/2)$ for $|x| < 2$. Its radius is 2 and both endpoints diverge.

Step 3: State where multiplication is justified. Both factors converge absolutely on $(-1, 1)$, so the Cauchy-product theorem proves equality there. Their rational formulas give

$$\frac{1}{1-x} \frac{1-x}{1-x/2} = \frac{1}{1-x/2}.$$

Cancellation removes the factor responsible for A 's boundary at 1. The coefficient series C consequently has radius 2, exceeding the smaller input radius. The theorem guarantees convergence on the common interior; it does not require the resulting radius to equal the smaller one.

Step 4: Interpret the additional points carefully. The new series gives $C(1) = 2$, $C(3/2) = 4$. At $x = 1$, A diverges and B sums to zero; an undefined series sum cannot be multiplied by zero to produce 2. At $x = 3/2$, A again diverges, although B converges to -2 . Thus neither value is a product of the two original series sums. They are valid values of the independently convergent product-coefficient series and its rational formula.