

Question 4

For an integer $N \geq 0$, define

$$S_N(x) = \sum_{n=0}^N x^n, \quad f(x) = \frac{1}{1-x}, \quad |x| < 1.$$

A student says that choosing a sufficiently large N makes this approximation uniformly accurate throughout the entire convergence interval.

Tasks

1. Derive the exact remainder $f - S_N$ from a finite geometric identity. State its sign for positive and negative x .
2. Find the least N guaranteeing absolute error at most 10^{-3} for every $x \in [-1/2, 1/2]$. State both the largest exponent and the number of retained terms.
3. Express the relative error $|f - S_N|/|f|$. For any fixed N , find a point in $(0, 1)$ where this error is exactly $1/2$.
4. Decide whether convergence is uniform on $(-1, 1)$, for either absolute or relative error, and justify the distinction from convergence at each fixed point. Plot S_2, S_5, S_{10} and f on $[0, 0.85]$.

Question 4 – Solution

Strategy. An exact remainder supplies a sharp interval bound and exposes what fails when points approach the boundary as the degree changes.

Step 1: Retain the exact geometric remainder. Multiplication gives $(1-x)S_N = 1 - x^{N+1}$. Therefore

$$f(x) - S_N(x) = \frac{x^{N+1}}{1-x}.$$

For $0 < x < 1$ it is positive. For $-1 < x < 0$, its sign is $(-1)^{N+1}$; at $x = 0$ it is zero. The denominator is positive throughout the interval.

Step 2: Find the least degree on the compact interval. For $|x| \leq 1/2$,

$$|f - S_N| \leq \frac{(1/2)^{N+1}}{1 - 1/2} = 2^{-N}.$$

Equality holds at $x = 1/2$, so this is the exact maximum, not merely a convenient estimate. Since $2^{-9} > 10^{-3}$ and $2^{-10} < 10^{-3}$, $N = 10$ is least. This retains 11 terms, including the constant.

Step 3: Select a moving point with fixed relative error. The relative error simplifies to $|x|^{N+1}$. Taking $x_N = 2^{-1/(N+1)}$ gives a point in $(0, 1)$ with relative error exactly $1/2$ for every $N \geq 0$. These points approach 1 as N increases.

Step 4: Distinguish fixed-point from uniform accuracy. At each fixed $|x| < 1$, both errors tend to zero. Yet for every fixed N , $x^{N+1}/(1-x) \rightarrow +\infty$ as $x \rightarrow 1^-$, so the supremum of the absolute error on $(-1, 1)$ is infinite. The supremum of relative error is 1. Neither error converges uniformly to zero on that whole interval. The plot shows slower approximation near the right boundary; on any fixed $[-r, r]$, $r < 1$, the bound $r^{N+1}/(1-r) \rightarrow 0$ does give uniform convergence.

