

Question 3

Let

$$F(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \quad G(x) = \int_0^x F(t) dt.$$

Use termwise differentiation and integration only where their hypotheses hold. No closed formula for F itself is required.

Tasks

1. Determine the radius and exact convergence interval of the series for F , including the type of convergence at each endpoint.
2. Obtain power series for F' and G . Determine each radius and interval, testing the endpoints afresh rather than copying them from F .
3. Sum the derivative series for $|x| < 1$ using the logarithmic series, treating $x = 0$ separately. Find its limit as $x \rightarrow -1^+$ and as $x \rightarrow 1^-$.
4. Prove F is continuous on $[-1, 1]$. Determine its one-sided endpoint slopes and explain why a finite endpoint value need not have a finite slope. Compare the three convergence intervals in a diagram.

Question 3 – Solution

Strategy. Differentiation and integration preserve the radius but can alter endpoints; continuity and the mean value theorem settle endpoint slopes.

Step 1: Check the original series. The ratio test gives $R = 1$. At $x = 1$ and $x = -1$, the absolute values sum to $\sum 1/n^2 < \infty$. Thus $I_F = [-1, 1]$, absolutely throughout. For $|x| > 1$ the terms do not tend to zero.

Step 2: Differentiate and integrate inside the radius. For $|x| < 1$,

$$F'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}, \quad G(x) = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n^2(n+1)}.$$

The constant in G is zero because $G(0) = 0$. Both new radii are 1. For the derivative series, $x = 1$ gives harmonic divergence; $x = -1$ gives conditional alternating convergence. Hence $I_{F'} = [-1, 1)$. The integrated series converges absolutely at both endpoints, so $I_G = [-1, 1]$. Here these interval labels refer to the displayed series, not an assumed endpoint differentiation rule.

Step 3: Sum the derivative in the open interval. Using $\sum_{n \geq 1} x^n/n = -\log(1-x)$ gives

$$F'(x) = \frac{-\log(1-x)}{x} \quad (0 < |x| < 1), \quad F'(0) = 1.$$

The value at zero follows from the constant term and also from the limit. Consequently $\lim_{x \rightarrow -1+} F'(x) = \log 2$, whereas $\lim_{x \rightarrow 1-} F'(x) = +\infty$.

Step 4: Justify what happens at the boundary. Since $|x^n/n^2| \leq 1/n^2$ on $[-1, 1]$, the uniform majorant test gives uniform convergence there and continuity of F . Apply the mean value theorem between an interior point and either endpoint. The intervening derivative then tends to the corresponding limit found above. Therefore

$$F'_+(-1) = \log 2, \quad \lim_{x \rightarrow 1-} \frac{F(1) - F(x)}{1 - x} = +\infty.$$

So the finite value $F(1)$ has no finite left derivative. Endpoint behavior must be proved separately even when the power-series radius is unchanged.

