

## Question 2

A power series with large gaps between nonzero coefficients is written as

$$L(x) = \sum_{k=0}^{\infty} 2^k x^{2^k}.$$

The first terms are  $x + 2x^2 + 4x^4 + 8x^8 + \cdots$ . Use the ordinary definition of a power series even though many coefficients vanish.

### Tasks

1. Write its coefficient rule in the form  $\sum_{n=0}^{\infty} a_n x^n$ . Explain why applying  $|a_{n+1}/a_n|$  to all consecutive coefficients is inappropriate.
2. Prove the exact radius using comparison when  $|x| < 1$  and the necessary term test when  $|x| > 1$ .
3. Test both endpoints, paying attention to the parity of  $2^k$ . Does the endpoint  $x = -1$  produce an alternating tail?
4. Let  $x_m = e^{-1/2^m}$  for integers  $m \geq 0$ . Use a single term to bound  $L(x_m)$  from below, and decide whether the sparse exponents make  $L$  bounded as  $x \rightarrow 1^-$ .

## Question 2 – Solution

**Strategy.** Treat the missing coefficients explicitly. A convergent positive comparison series controls the interior, while selected large terms control boundary growth.

**Step 1: Identify the coefficients.** We have  $a_n = n$  when  $n = 2^k$  for an integer  $k \geq 0$ , and  $a_n = 0$  otherwise, including  $n = 0$ . Consecutive coefficient ratios repeatedly have zero denominators, so the usual coefficient-ratio shortcut is unavailable. This does not prevent other convergence tests from determining the radius.

**Step 2: Establish the radius directly.** For  $r = |x| < 1$ ,

$$\sum_{k=0}^{\infty} 2^k r^{2^k} \leq \sum_{n=1}^{\infty} n r^n = \frac{r}{(1-r)^2} < \infty.$$

Thus the series converges absolutely throughout  $|x| < 1$ . If  $|x| > 1$ , the magnitudes  $2^k |x|^{2^k}$  do not tend to zero; indeed they grow without bound. Therefore  $\boxed{R = 1}$ . The comparison sum follows by differentiating the geometric series inside its radius.

**Step 3: Test parity at the endpoints.** At  $x = 1$ , the terms are  $2^k$  and fail the term test. At  $x = -1$ , the first term is  $-1$ , but every term with  $k \geq 1$  is  $+2^k$ , since  $2^k$  is even. There is no alternating tail. Both endpoint series diverge, so  $\boxed{I = (-1, 1)}$ , with absolute convergence at every point of this interval.

**Step 4: Prove unbounded growth despite the gaps.** For  $0 < x_m < 1$ , every term is positive. Keeping just the term indexed by  $k = m$  gives

$$\boxed{L(x_m) \geq 2^m x_m^{2^m} = 2^m / e \rightarrow \infty.}$$

Also  $x_m \uparrow 1$ . Thus  $L$  is not bounded near 1. In fact the sum is increasing on  $[0, 1)$  because each term is increasing, so this sequence of lower bounds proves  $L(x) \rightarrow +\infty$  as  $x \rightarrow 1^-$ . Sparse powers do not offset the growth of the coefficients and their near-boundary terms.