

## Question 1

Consider the real power series

$$F(x) = \sum_{n=1}^{\infty} \frac{(x-2)^n}{n3^n}.$$

A student reports a radius of convergence and assumes that this alone specifies where the series represents a finite function.

### Tasks

1. Identify the center and radius of convergence. Explain what the ratio test says inside and outside the radius, and what it leaves undecided.
2. Test both boundary points separately. Give the exact interval of convergence and distinguish absolute from conditional convergence.
3. Derive a closed formula for the sum inside the open interval by integrating a geometric series. Include the constant fixed by the value at the center.
4. Determine the sum at the included boundary point without substituting into an unjustified endpoint identity. Sketch the convergence interval and explain why the excluded endpoint behaves differently.

## Question 1 – Solution

**Strategy.** Normalize the displacement from the center, test the endpoints as numerical series, then justify any boundary value independently.

**Step 1: Apply the ratio test away from the center.** Write  $z = (x - 2)/3$ . For  $z \neq 0$ , the ratio of successive term magnitudes is  $|z|n/(n+1) \rightarrow |z|$ . Thus the center is 2 and  $R = 3$ . The series converges absolutely for  $|z| < 1$  and diverges for  $|z| > 1$ . At the center all terms vanish. The ratio test is inconclusive at  $z = \pm 1$ .

**Step 2: Resolve the two endpoints.** At  $x = -1$ , the series is  $\sum_{n \geq 1} (-1)^n/n$ , which converges by alternation but not absolutely. At  $x = 5$ , it is the divergent harmonic series. Therefore

$$I = [-1, 5), \quad \text{absolute on } (-1, 5), \quad \text{conditional only at } -1.$$

A radius specifies distances, not whether either boundary point is included.

**Step 3: Integrate the geometric series.** On any compact subinterval of  $(-1, 1)$ , the geometric series converges uniformly, so integration from 0 to  $z$  gives

$$\sum_{n=1}^{\infty} \frac{z^n}{n} = \int_0^z \frac{du}{1-u} = -\log(1-z).$$

This also holds for negative  $z$  with the usual oriented integral. Both sides vanish at  $z = 0$ , fixing the constant. Hence  $F(x) = -\log(1 - (x - 2)/3)$  for  $-1 < x < 5$ .

**Step 4: Establish the boundary value.** The finite geometric identity gives  $\sum_{k=0}^{N-1} (-u)^k = (1 - (-u)^N)/(1 + u)$ . Integrating over  $[0, 1]$ , its difference from  $\log 2$  has magnitude at most  $\int_0^1 u^N du = 1/(N+1) \rightarrow 0$ . Negating the resulting alternating harmonic sum proves  $F(-1) = -\log 2$ . At  $x = 5$  there is no alternation and harmonic divergence persists; the interior logarithm also tends to  $+\infty$  there.

