

Question 10

For $\varepsilon \geq 0$, consider

$$A_\varepsilon = \begin{pmatrix} -1 & \varepsilon & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad X' = A_\varepsilon X, \quad X(0) = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

A matrix is diagonalizable if it has a basis of eigenvectors. For nonzero solutions, compare their normalized directions $X/\|X\|$.

Tasks

1. Find the eigenvalues and eigenspaces. For which ε is the matrix diagonalizable?
2. Find the complete solution for arbitrary initial data, including the polynomial term when present.
3. Identify all forward-bounded initial states. Within that set, classify the limiting forward directions for every nonzero initial state.
4. For initial $(0, 1, 0)^T$, compare the iterated limits of $X_\varepsilon/\|X_\varepsilon\|$ as $t \rightarrow \infty$ and $\varepsilon \downarrow 0$, in both orders. Explain what changes even though the eigenvalues do not.

Question 10 – Solution

Strategy. A repeated stable eigenvalue can have a two-dimensional eigenspace or a defective chain; the limiting direction detects that difference.

Step 1: Determine the eigenspaces. The eigenvalues are -1 (multiplicity two) and 2 . For -1 , $z = 0$ and $\varepsilon y = 0$. Thus its eigenspace is the whole plane $z = 0$ when $\varepsilon = 0$, but only the x -axis when $\varepsilon > 0$. The 2 eigenspace is always $\text{span}(1, 0, 1)^T$. There is a full eigenbasis exactly when $\boxed{\varepsilon = 0}$; this diagonalizable three-dimensional matrix is not a scalar matrix.

Step 2: Solve the coupled components. First $z = re^{2t}$ and $y = qe^{-t}$. Then $(e^t x)' = \varepsilon q + 3re^{3t}$, giving

$$X = e^{-t} \begin{pmatrix} p - r + \varepsilon q t \\ q \\ 0 \end{pmatrix} + re^{2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

This satisfies every component and recovers $(p, q, r)^T$ at zero.

Step 3: Classify the stable-plane directions. Forward boundedness holds exactly when $r = 0$. On that plane all states decay. For $\varepsilon = 0$, the nonzero direction stays constantly $(p, q, 0)^T / \sqrt{p^2 + q^2}$. For $\varepsilon > 0$ and $q \neq 0$, the normalized direction tends to $(\text{sgn } q, 0, 0)^T$. If $q = 0$, $p \neq 0$, it is constantly $(\text{sgn } p, 0, 0)^T$. The zero solution has no normalized direction.

Step 4: Compare the two orders of limits. For the specified initial state, the normalized vector is $(\varepsilon t, 1, 0)^T / \sqrt{\varepsilon^2 t^2 + 1}$. Taking $t \rightarrow \infty$ first with $\varepsilon > 0$, then $\varepsilon \downarrow 0$, gives $\boxed{(1, 0, 0)^T}$. Reversing the operations gives $\boxed{(0, 1, 0)^T}$. At every fixed finite time the solution depends continuously on ε , but long-time direction is not uniform near zero. The eigenvalues stay fixed while the eigenspace dimension changes.