

Question 4

For $\varepsilon > 0$, let

$$X'_\varepsilon = \begin{pmatrix} -1 & 1 \\ 0 & -1 - \varepsilon \end{pmatrix} X_\varepsilon, \quad X_\varepsilon(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Study the limit as the two distinct eigenvalues merge. Let X_0 denote the solution of the same IVP with $\varepsilon = 0$.

Tasks

1. Solve the IVP for $\varepsilon > 0$ by using the distinct real modes.
2. Find the limit at each fixed time as $\varepsilon \downarrow 0$ and verify the resulting repeated-root solution.
3. For $0 \leq t \leq T$, prove bounds on $|x_\varepsilon - x_0|$ and $|y_\varepsilon - y_0|$ that tend uniformly to zero as $\varepsilon \downarrow 0$. You may use $0 \leq 1 - e^{-u} \leq u$ for $u \geq 0$.
4. At $t = 1/\varepsilon$, compare x_ε/x_0 with 1. Explain how coefficients that diverge and a relative error that persists can coexist with the fixed-time limit.

Question 4 – Solution

Strategy. Keep the difference of nearby exponentials together; an integral representation exposes the limit without subtracting divergent pieces.

Step 1: Solve before the roots merge. Eigenvectors are $(1, 0)^T$ for -1 and $(1, -\varepsilon)^T$ for $-1 - \varepsilon$. Decomposition of $(0, 1)^T$ gives coefficients $1/\varepsilon, -1/\varepsilon$, hence

$$x_\varepsilon = e^{-t} \frac{1 - e^{-\varepsilon t}}{\varepsilon}, \quad y_\varepsilon = e^{-(1+\varepsilon)t}.$$

Step 2: Pass to the repeated-root solution. Since $(1 - e^{-\varepsilon t})/\varepsilon = \int_0^t e^{-\varepsilon s} ds$, its limit at each fixed real t is t . Thus $X_0 = e^{-t}(t, 1)^T$. Direct differentiation gives $x'_0 = -x_0 + y_0$, $y'_0 = -y_0$ and the stated initial data. The polynomial factor is the limit of the combined modes.

Step 3: Prove uniform finite-interval control. For $0 \leq t \leq T$, the supplied inequality yields

$$0 \leq x_0 - x_\varepsilon = e^{-t} \int_0^t (1 - e^{-\varepsilon s}) ds \leq \frac{1}{2} \varepsilon t^2 e^{-t} \leq \frac{1}{2} \varepsilon T^2.$$

Similarly $0 \leq y_0 - y_\varepsilon = e^{-t}(1 - e^{-\varepsilon t}) \leq \varepsilon t e^{-t} \leq \varepsilon T$. Both bounds tend uniformly to zero on every fixed finite interval.

Step 4: Separate relative and absolute effects. At $t = 1/\varepsilon$, $x_\varepsilon/x_0 = 1 - e^{-1}$, so the relative discrepancy is e^{-1} , independent of ε . This time moves to infinity, and $x_0 = \varepsilon^{-1} e^{-1/\varepsilon} \rightarrow 0$, so the absolute discrepancy still vanishes there. The separate modal coefficients diverge, but their exponential contributions cancel to a finite limit. Neither divergent coefficients nor this moving-time relative error contradicts the proved finite-interval convergence.

