

Question 3

Consider

$$X' = AX, \quad A = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

For a repeated eigenvalue λ , a length-two chain means vectors v, w with $(A - \lambda I)v = 0$ and $(A - \lambda I)w = v \neq 0$.

Tasks

1. Find the eigenvalue, its eigenspace, and a chain with $v = (1, 1)^T$.
2. Derive and verify two independent real solutions using this chain. Explain why $te^{\lambda t}v$ alone fails.
3. Solve the IVP and verify the initial state and derivative.
4. Find every generalized vector w compatible with the fixed v . Explain how changing w changes the coefficients without changing the IVP solution.

Question 3 – Solution

Strategy. The generalized vector cancels the extra derivative introduced by the polynomial factor.

Step 1: Build a chain explicitly. The characteristic polynomial is $(\lambda+2)^2$. Let $N = A+2I = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$; then $N^2 = 0$. Its kernel is $\text{span}(1, 1)^T$. With $v = (1, 1)^T$, choose $w = (1, 0)^T$, since $Nw = v$. The two vectors are independent.

Step 2: Verify the missing solution. Set $U = e^{-2t}v$ and $V = e^{-2t}(w + tv)$. Using $Nv = 0$, $Nw = v$, differentiation gives $U' = AU$ and $V' = AV$. Their determinant is $e^{-4t} \det(v, w) = -e^{-4t} \neq 0$, so they form a complete real basis. For $Z = te^{-2t}v$, however, $Z' - AZ = e^{-2t}v \neq 0$: the polynomial term by itself is not a solution.

Step 3: Determine the IVP coefficients. Since $(2, 1)^T = v + w$, the required solution is $U + V$:

$$X = e^{-2t} \begin{pmatrix} 2+t \\ 1+t \end{pmatrix}.$$

At zero it is $(2, 1)^T$, and its derivative is $(-3, -1)^T$, agreeing with $A(2, 1)^T$. The mode calculation verifies the equation for every time, not just at the initial point.

Step 4: Account for chain freedom. The equation $Nw = v$ means $w_1 - w_2 = 1$, so every choice is $w_c = (1 + c, c)^T = w + cv$, $c \in \mathbb{R}$. The new second solution is $V_c = V + cU$. Hence the same IVP becomes $X = (1 - c)U + V_c$. In general a combination $aU + bV$ becomes $(a - bc)U + bV_c$. The basis changes, but its span and every physical solution remain the same. Scaling or shifting a chain must be accompanied by the corresponding coefficient change.