

**Question 2**

Let

$$X' = NX, \quad N = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Both eigenvalues are zero. A matrix is nilpotent if some positive power of it is zero; investigate the effect on the actual motion.

**Tasks**

1. Compute  $N^2$ , find the entire equilibrium set, and determine its dimension.
2. Find the normalized evolution matrix and the solution from arbitrary  $(p, q)^T$ . Verify the evolution law directly.
3. For initial data outside the equilibrium line, find the phase trajectory and its time direction. Can it reach that line at a finite or infinite forward time?
4. Classify all forward-bounded solutions. Decide whether the origin is stable in the sense that sufficiently small initial states remain small for all  $t \geq 0$ .

## Question 2 – Solution

**Strategy.** A zero eigenvalue can preserve an equilibrium line while a nilpotent part creates unbounded drift parallel to it.

**Step 1: Identify the nilpotent part and equilibria.** Direct multiplication gives  $N^2 = 0$  but  $N \neq 0$ . The equation  $NX = 0$  requires  $x = y$ , so the equilibria form the one-dimensional line  $L = \{(s, s) : s \in \mathbb{R}\}$ . The double zero eigenvalue has only one independent eigendirection.

**Step 2: Construct the complete flow.** The matrix  $F(t) = I + tN$  has  $F(0) = I$  and  $F' = N = NF$ , since  $N^2 = 0$ . Thus

$$X(t) = \begin{pmatrix} p \\ q \end{pmatrix} + t(p - q) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Also  $(I + tN)(I + sN) = I + (t + s)N$ , verifying the evolution law and  $F(t)^{-1} = F(-t)$ . All real initial states and all real times are allowed.

**Step 3: Find the drifting straight lines.** Put  $d = p - q$ . If  $d \neq 0$ , then  $x - y = d$  for every time and the velocity is the constant vector  $d(1, 1)^T$ . Thus the full orbit is the entire line  $x - y = d$ , traversed up/right for  $d > 0$  and down/left for  $d < 0$ . Its distance to  $L$  stays  $|d|/\sqrt{2} > 0$ , so it neither reaches nor approaches the equilibrium line. If  $d = 0$ , the orbit is one fixed point, not the entire line  $L$ .

**Step 4: Test boundedness and stability.** Exactly  $p = q$  gives a forward-bounded solution; all other solutions drift without bound. The origin is unstable: initial  $(\varepsilon, 0)$  can be arbitrarily small, yet evolves to  $(\varepsilon(1 + t), \varepsilon t)$ , eventually leaving any fixed neighborhood. Zero real parts alone do not ensure boundedness or stability when a nonzero nilpotent part is present. The dashed line in the figure consists of separate equilibria.

Solid: drifting orbits; dashed: equilibrium line

