

Question 1

Compare the systems $X' = AX$ and $X' = BX$, where

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}.$$

Both have the characteristic polynomial $(\lambda + 2)^2$.

Tasks

1. Find the eigenspace of each matrix and explain why the repeated characteristic root does not determine the number of independent eigenvectors.
2. Write the complete real solution of each system for initial state $(p, q)^T$. Verify any term containing t .
3. For the B system starting at $(0, 1)$, eliminate time with the correct domain. Find the maximum of $x(t)$ on $t \geq 0$.
4. Compare the two systems' phase trajectories and forward limits. Does every repeated eigenvalue require a polynomial factor in every solution?

Question 1 – Solution

Strategy. The eigenspace dimension decides whether a repeated root already supplies a full basis or needs another solution.

Step 1: Distinguish the eigenspaces. For A , every nonzero vector is a -2 eigenvector, so the eigenspace has dimension two. For B , $(B + 2I)(x, y)^T = (y, 0)^T$, so its eigenspace is $\text{span}(1, 0)^T$, of dimension one. Algebraic multiplicity is two in both cases; the geometric multiplicities differ.

Step 2: Recover both complete families. For initial (p, q) ,

$$X_A = e^{-2t} \begin{pmatrix} p \\ q \end{pmatrix}, \quad X_B = e^{-2t} \begin{pmatrix} p + qt \\ q \end{pmatrix}.$$

For B , first solve $y' = -2y$, then $(e^{2t}x)' = q$. This derives every solution, and direct differentiation verifies the qt term. In particular $te^{-2t}(1, 0)^T$ alone is not the missing solution; the accompanying $e^{-2t}(0, 1)^T$ is essential.

Step 3: Analyze the selected orbit. For B with $(p, q) = (0, 1)$, $x = te^{-2t}$ and $y = e^{-2t} > 0$. Eliminating t gives

$$x = -\frac{1}{2}y \ln y, \quad y > 0.$$

Forward time decreases y . On $t \geq 0$, $x' = e^{-2t}(1 - 2t)$ changes sign at $t = 1/2$, giving $x_{\max} = 1/(2e)$ at $y = e^{-1}$.

Step 4: Interpret multiplicity and geometry. Every nonzero A trajectory is a straight ray toward zero. For B , only the eigenline $y = 0$ gives straight rays; otherwise $x/y = p/q + t$ changes, giving curved trajectories. All solutions of both systems decay. A repeated root needs no polynomial factor when there is a full eigenbasis; even in the defective system, eigenline initial data have $q = 0$ and no t term. The figure shows the selected B orbit, not a generic ray.

$x = -\frac{1}{2}y \ln y$; arrows show increasing time

