

Question 10

For $\gamma > 0$, consider the four-dimensional system

$$x'_1 = -x_2, \quad x'_2 = x_1, \quad x'_3 = -\gamma x_4, \quad x'_4 = \gamma x_3.$$

A state is periodic if there is $T > 0$ with $X(t+T) = X(t)$ for all real t . A constant solution has no least positive period.

Tasks

1. Find the eigenvalues and solve the IVP $X(0) = (1, 0, 1, 0)^T$. Describe its projections onto the two coordinate planes.
2. For $\gamma = \sqrt{2}$, prove that this full state is not periodic, even though both planar projections are periodic.
3. Still with $\gamma = \sqrt{2}$, classify exactly which arbitrary initial states give periodic solutions, and give their least periods when nonconstant.
4. Replace γ by $3/2$. Find the least period for the stated IVP and explain why purely imaginary eigenvalues alone do not guarantee a common period in higher dimensions.

Question 10 – Solution

Strategy. A full state repeats only when every active rotating block repeats at the same time.

Step 1: Solve each oscillatory block. The eigenvalues are $\pm i$ and $\pm i\gamma$. For the stated initial data,

$$X(t) = (\cos t, \sin t, \cos \gamma t, \sin \gamma t)^T.$$

The two projections are unit circles with respective least periods 2π and $2\pi/\gamma$. Their closure as plane curves alone does not settle repetition of the combined state.

Step 2: Exclude a common period for irrational frequency ratio. At $\gamma = \sqrt{2}$, any period must satisfy $T = 2\pi m$ and $\sqrt{2}T = 2\pi n$ for positive integers m, n . This would imply $\sqrt{2} = n/m$, impossible. Hence the four-dimensional state is not periodic. Its norm nevertheless remains $\sqrt{2}$, so the obstruction is incompatible timing, not growth or damping.

Step 3: Classify all initial states in the irrational case. Each initially zero two-component block remains zero. A nonzero first block repeats exactly at integer multiples of 2π ; a nonzero second block repeats exactly at integer multiples of $2\pi/\sqrt{2}$. Thus precisely the states with at least one block zero give periodic solutions. If only the first block is nonzero, the least period is 2π ; if only the second is nonzero, it is $\sqrt{2}\pi$. If both vanish, the solution is constant. If both are nonzero, the same irrationality argument rules out every positive period, regardless of their initial phases.

Step 4: Recover a common period for rational frequency ratio. With $\gamma = 3/2$ and both blocks active, $T = 2\pi m$ must also satisfy $3m/2 \in \mathbb{Z}$. The least positive choice is $m = 2$, giving $T = 4\pi$. In these independent rotation systems all states are bounded, but common periodicity requires commensurate active frequencies. Two planar centers therefore need not combine into a periodic full state; purely imaginary eigenvalues alone do not provide the missing frequency ratio.