

Question 8

For a nonzero real frequency ω , consider

$$X' = \begin{pmatrix} -1 & -\omega \\ \omega & -1 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

An instrument records only selected times. At every integer $n \geq 0$ it reports $X(n) = e^{-n}(1, 0)^T$. Assume the measurements are exact.

Tasks

1. Solve the IVP and determine every nonzero frequency consistent with all the integer-time measurements.
2. Can these measurements determine rotation direction or the number of revolutions per unit time? Explain why the observed points do not imply a straight-line continuous trajectory.
3. An additional measurement is $X(1/4) = e^{-1/4}(0, 1)^T$. Find every frequency still allowed, including negative frequencies.
4. If one also knows $0 < \omega < 4\pi$, recover the frequency uniquely. Exhibit two different frequencies that fit both sets of measurements when that extra bound is absent.

Question 8 – Solution

Strategy. Sampling observes angles only modulo complete revolutions; additional information is needed to recover the continuous frequency.

Step 1: Solve and impose integer samples. The eigenvalues are $-1 \pm i\omega$, and

$$X(t) = e^{-t} \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix}.$$

The measurement at $n = 1$ requires $\omega = 2\pi k$, $k \in \mathbb{Z}$. Every such value satisfies all integer samples; the nonzero-frequency assumption excludes $k = 0$.

Step 2: Identify what the samples conceal. The sign of k gives counterclockwise or clockwise rotation, and $|k|$ gives revolutions per unit time. Neither is fixed by the integer samples. For every permitted nonzero frequency the continuous angle changes and the radius decays, tracing a spiral. The sampled points all lie on one ray because whole turns occur between them, not because that ray is an invariant trajectory.

Step 3: Use the quarter-time observation. The additional angle condition is $\omega/4 = \pi/2 + 2\pi m$. Thus

$$\omega = 2\pi(1 + 4m), \quad m \in \mathbb{Z}.$$

These values already satisfy the integer constraints. Negative possibilities remain, for example $m = -1$ gives $\omega = -6\pi$. No zero frequency occurs in this family.

Step 4: Use the external frequency bound. Under $0 < \omega < 4\pi$, only $m = 0$ remains, giving $\omega = 2\pi$. Without that bound, 2π and -6π both fit all stated observations, despite opposite directions and different speeds. The figure plots $e^t y(t)$ for these two solutions: they agree at the recorded times but differ between them. The vertical variable is the rescaled component, not the radius or a phase-plane coordinate.

