

Question 6

Let

$$A = \begin{pmatrix} 2 & -5 \\ 1 & 0 \end{pmatrix}, \quad B = A - I.$$

Find the normalized evolution matrix for $X' = AX$ without selecting or normalizing complex eigenvectors. A normalized evolution matrix F satisfies $F' = AF$ and $F(0) = I$.

Tasks

1. Find the eigenvalues and prove a simple identity for B^2 .
2. Construct a real formula for $F(t)$ using that identity, and verify both defining conditions.
3. Solve the IVP $X(0) = (1, 1)^T$. Compute $F(\pi)$ and explain why a complete oscillation does not produce a periodic nonzero solution.
4. Prove the multiplication law $F(t + s) = F(t)F(s)$, find $F(t)^{-1}$, and compute its determinant. Does the determinant alone determine how every vector length changes?

Question 6 – Solution

Strategy. Remove the real part of the spectrum; the remaining matrix acts algebraically like multiplication by an imaginary number.

Step 1: Isolate the oscillatory matrix. The characteristic polynomial is $\lambda^2 - 2\lambda + 5$, with roots $1 \pm 2i$. Here $B = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}$ and direct multiplication gives $B^2 = -4I$.

Step 2: Construct a real evolution formula. Set

$$F(t) = e^t \left(I \cos 2t + \frac{1}{2} B \sin 2t \right).$$

Its value at zero is I . Differentiating the bracket gives $-2I \sin 2t + B \cos 2t$, which equals B times that bracket because $B^2 = -4I$. The product rule therefore yields $F' = (I + B)F = AF$. Uniqueness verifies that this formula evolves every initial state.

Step 3: Solve and inspect one oscillation. Since $B(1, 1)^T = (-4, 0)^T$,

$$X = e^t \begin{pmatrix} \cos 2t - 2 \sin 2t \\ \cos 2t \end{pmatrix}.$$

The initial state is $(1, 1)^T$ and initial derivative $(-3, 1)^T = A(1, 1)^T$. Also $F(\pi) = e^\pi I$: a full trigonometric cycle multiplies the state by e^π . More generally the periodic bracket and its inverse are bounded, so every nonzero solution grows without bound forward in time. Thus there is no nonzero periodic solution.

Step 4: Verify the evolution algebra. Multiplying two brackets and using $B^2 = -4I$ gives the sine and cosine addition formulas, hence $F(t + s) = F(t)F(s)$. Therefore $F(t)^{-1} = F(-t)$. For scalars c, s , $\det(cI + sB/2) = c^2 + s^2$, so $\det F(t) = e^{2t}$. This controls area, not every vector length: at the state $(1, 1)$ the squared-length derivative is $2(1, 1) \cdot (-3, 1) = -4 < 0$, despite increasing area. Shape distortion and eventual growth can coexist.