

**Question 4**

For  $a \in \mathbb{R}$ , consider

$$X' = \begin{pmatrix} a & -3 \\ 3 & a \end{pmatrix} X.$$

For  $r > 0$ , define the positive-axis return map  $P_a(r)$  as the radius the solution starting at  $(r, 0)$  has on its first later return to the positive  $x$ -axis. A return to this ray need not be a return to the same point.

**Tasks**

1. Find the eigenvalues and the solution from an arbitrary nonzero initial state in polar form.
2. Classify the origin for  $a < 0$ ,  $a = 0$ , and  $a > 0$ . Determine exactly when a nonzero solution is periodic.
3. Find the first return time, the map  $P_a$ , and all its fixed radii in  $r > 0$ . Explain their relation to periodic trajectories.
4. An experiment finds  $P_a(r) = r/2$  for one nonzero radius. Recover  $a$ , and find the radius after  $n$  complete turns for every integer  $n \geq 0$ .

## Question 4 – Solution

**Strategy.** The real part sets radial growth and the imaginary part sets turning time; a ray return becomes periodic only when the radius is restored.

**Step 1: Read radial and angular evolution.** The eigenvalues are  $a \pm 3i$ . For initial radius  $r_0 > 0$  and angle  $\theta_0$ , the solution has

$$r(t) = r_0 e^{at}, \quad \theta(t) = \theta_0 + 3t.$$

Equivalently  $X(t) = e^{at} R(3t) X(0)$ , where  $R(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$ . The only equilibrium is zero because the determinant is  $a^2 + 9 > 0$ .

**Step 2: Classify all parameter cases.** For  $a < 0$  the origin is a stable spiral; for  $a > 0$  it is an unstable spiral. At  $a = 0$  it is a center: every nonzero state follows a circle with least period  $2\pi/3$ . For  $a \neq 0$ , the radius is strictly monotone, so no nonzero solution is periodic. Zero remains constant and has no least positive period.

**Step 3: Compute the return map.** The first return to the positive ray takes  $T = 2\pi/3$ , independent of  $r$  and  $a$ . Consequently

$$P_a(r) = e^{2\pi a/3} r.$$

For  $r > 0$ , the fixed-point equation holds for every radius if  $a = 0$  and for none if  $a \neq 0$ . These fixed radii are exactly the nonzero periodic trajectories. Returning to an angle alone is insufficient.

**Step 4: Recover damping from one revolution.** The measurement gives  $e^{2\pi a/3} = 1/2$ , hence

$$a = -\frac{3 \ln 2}{2\pi}, \quad r_n = r_0 2^{-n}.$$

The coefficient  $a$  is unique because the real exponential is one-to-one. The same multiplier applies to every radius by linearity; a single nonzero measurement therefore determines the damping rate in this specified family.