

Question 2

Consider

$$x' = 4y, \quad y' = -x, \quad (x(0), y(0)) = (2, 0).$$

A student sees eigenvalues $\pm 2i$ and claims that the trajectory must be a circle rotating with constant Euclidean angular speed 2.

Tasks

1. Find the eigenvalues and solve the IVP in real form.
2. Find a conserved positive quadratic expression and determine the complete phase curve. Locate its intercepts.
3. Find the least positive period and the direction of traversal. Identify coordinates in which the motion is a uniform circular rotation.
4. Compute the original Euclidean angular velocity $\theta' = (xy' - yx')/(x^2 + y^2)$. Find its range on the orbit and evaluate the student's claim.

Question 2 – Solution

Strategy. A linear coordinate change can turn a circle into an ellipse while preserving elapsed time and period.

Step 1: Solve the oscillatory IVP. The characteristic polynomial is $\lambda^2 + 4$, giving $\pm 2i$. From $x'' = -4x$ and $x'(0) = 0$,

$$x = 2 \cos 2t, \quad y = -\sin 2t.$$

Differentiation verifies both first-order equations and the initial data.

Step 2: Recover the invariant ellipse. The derivative of $H = x^2 + 4y^2$ is $8xy - 8xy = 0$. Thus the orbit is $x^2 + 4y^2 = 4$, with intercepts $(\pm 2, 0)$ and $(0, \pm 1)$. The parametrization covers the whole ellipse. Its unequal semiaxes must remain unequal in an equal-scale drawing; it is not a circle in the original Euclidean coordinates.

Step 3: Determine period and orientation. The least positive period is π . At $(2, 0)$ the velocity is $(0, -2)$, so traversal is clockwise. The coordinates $u = x$, $v = 2y$ satisfy $u' = 2v$, $v' = -2u$ and $u^2 + v^2 = 4$. In that transformed plane the angular velocity is constantly -2 . The invertible transformation preserves the least period.

Step 4: Compute the actual angular speed. In the original plane,

$$\theta' = -\frac{x^2 + 4y^2}{x^2 + y^2} = -\frac{4}{x^2 + y^2}.$$

On the ellipse, $1 \leq x^2 + y^2 \leq 4$, so $-4 \leq \theta' \leq -1$. It equals -1 at the horizontal intercepts and -4 at the vertical ones. Thus imaginary eigenvalues determine a temporal frequency, but do not force a Euclidean circle or constant Euclidean angular speed. The average signed angular velocity over a complete traversal is nevertheless -2 .

