

Question 9

Let

$$A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}, \quad P = A + 2I, \quad Q = -(A + I).$$

Here a projection means a matrix R with $R^2 = R$; it need not be an orthogonal projection. Develop a solution formula without choosing normalized eigenvectors.

Tasks

1. Find the eigenvalues and verify $P + Q = I$, $P^2 = P$, $Q^2 = Q$, and $PQ = QP = 0$.
2. Identify the range and kernel of each projection. Are these projections orthogonal in the Euclidean plane?
3. Prove that $F(t) = e^{-t}P + e^{-2t}Q$ satisfies $F' = AF$ and $F(0) = I$. Use it to solve the IVP $X(0) = (1, 0)^T$.
4. Prove $F(t+s) = F(t)F(s)$ and find its inverse. Explain why these identities hold for every pair of real times here.

Question 9 – Solution

Strategy. Polynomial projections isolate the two real eigenspaces and make the evolution law a scalar calculation on each one.

Step 1: Verify complementary projections. The characteristic polynomial is $(\lambda+1)(\lambda+2)$, and direct multiplication gives $A^2 + 3A + 2I = 0$. Here $P = \begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix}$ and $Q = \begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}$. Thus $P + Q = I$, $P^2 = P$, $Q^2 = Q$, and $PQ = QP = 0$. For example $PQ = -(A + 2I)(A + I) = 0$; the other identities follow by multiplication or from $Q = I - P$.

Step 2: Identify the oblique directions. The range of P is $\text{span}(2, 3)^T$, the -1 eigenspace; its kernel is $\text{span}(1, 1)^T$, the -2 eigenspace. For Q the range and kernel are interchanged. These two directions have nonzero dot product 5, so neither projection is orthogonal. The matrices are also visibly nonsymmetric.

Step 3: Construct and verify evolution. The same polynomial identity gives $AP = -P$, $AQ = -2Q$. Consequently $F' = AF$ and $F(0) = P + Q = I$. For arbitrary initial vector X_0 , $F(t)X_0$ solves the IVP; constant linear-system uniqueness gives the complete solution. In particular

$$X = e^{-t} \begin{pmatrix} -2 \\ -3 \end{pmatrix} + e^{-2t} \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Its initial state is $(1, 0)^T$ and its initial derivative $(-4, -3)^T = A(1, 0)^T$.

Step 4: Multiply by modes. All mixed products vanish, so

$$F(t)F(s) = e^{-(t+s)}P + e^{-2(t+s)}Q = F(t+s).$$

Taking $s = -t$ yields $F(t)^{-1} = F(-t) = e^tP + e^{2t}Q$. No finite real time makes either scalar exponential vanish. This is the normalized evolution of one constant matrix; the fixed complementary projections justify the multiplication rule for every $s, t \in \mathbb{R}$. No orthogonality or eigenvector normalization was used.