

Question 8

Let

$$X' = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix} X, \quad x(0) = 1, \quad y(T) = b, \quad T > 0.$$

Only the initial value $q = y(0)$ is unknown. These conditions concern different times; they are not a complete initial state at one time.

Tasks

1. Find the real eigenpairs and write the solution in terms of q .
2. Derive the equation that q must satisfy. Identify every positive time at which its coefficient vanishes.
3. At the exceptional time, classify all values of b by whether there are zero, one, or infinitely many solutions. Explain why uniqueness for an IVP is not contradicted.
4. For other times, give q explicitly and its change under a measurement error $b \mapsto b + \delta$. Describe what happens near the exceptional time.

Question 8 – Solution

Strategy. Two real exponential modes can cancel in an endpoint observation even when the full state remains uniquely determined by its initial data.

Step 1: Find the two modes. The characteristic polynomial is $\lambda^2 - 1$. Eigenvectors are $(1, 1)^T$ for 1 and $(1, 2)^T$ for -1 . The data $(1, q)^T$ give coefficients $2 - q$ and $q - 1$, so

$$X = (2 - q)e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (q - 1)e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Step 2: Locate the singular observation time. The endpoint condition becomes

$$b = 2e^T - 2e^{-T} + q(2e^{-T} - e^T).$$

The coefficient of q vanishes exactly when $e^{2T} = 2$, hence $T_* = \frac{1}{2} \ln 2$, the unique positive exceptional time.

Step 3: Classify compatibility at that time. At T_* , the equation reduces to $b = \sqrt{2}$. If $b = \sqrt{2}$, every real q gives a solution; otherwise none does. There is never exactly one at this time. Distinct q specify distinct initial states. In fact $x(T_*) = (3 - q)/\sqrt{2}$, so the full states remain different even though their second components agree. Each individual IVP still has its unique solution. The figure shows three such second components.

Step 4: Quantify sensitivity away from the exception. For $T \neq T_*$ there is exactly one solution, with

$$q = \frac{b - 2e^T + 2e^{-T}}{2e^{-T} - e^T}, \quad \Delta q = \frac{\delta}{2e^{-T} - e^T}.$$

The denominator tends to zero as $T \rightarrow T_*$, so the absolute amplification factor diverges. This is a poorly conditioned recovery of missing initial data, not loss of uniqueness for a fully specified IVP. For a fixed nonzero δ , the recovered initial error can become arbitrarily large.

Solid $q = 0$; dashed $q = 1$; dotted $q = 2$

