

Question 5

For a real parameter $-1 < a < 1$, consider

$$X' = A_a X, \quad A_a = \frac{1}{2} \begin{pmatrix} a-2 & a+2 \\ a+2 & a-2 \end{pmatrix}.$$

Treat the zero-eigenvalue case explicitly; do not assume that every equilibrium is isolated.

Tasks

1. Find the eigenvalues, an eigenbasis independent of a , and the solution with arbitrary initial state $(p, q)^T$.
2. Classify the origin for $a < 0$ and $a > 0$, and identify the initial states that converge to it in each case.
3. At $a = 0$, find every equilibrium and the limiting state of every solution. Is the origin attracting all nearby states?
4. For initial state $(2, 0)^T$, compare $\lim_{a \uparrow 0} \lim_{t \rightarrow \infty} X_a(t)$ with $\lim_{t \rightarrow \infty} \lim_{a \uparrow 0} X_a(t)$. Explain the difference.

Question 5 – Solution

Strategy. Keep the parameter-dependent mode visible; its decay disappears at the zero eigenvalue.

Step 1: Use fixed eigendirections. The vectors $(1, 1)^T$ and $(1, -1)^T$ have eigenvalues a and -2 . They remain independent, and their eigenvalues are distinct throughout the stated parameter interval. Setting $m = (p + q)/2$, $d = (p - q)/2$ gives

$$X_a(t) = me^{at} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + de^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This formula satisfies the system and the initial data for every allowed a .

Step 2: Classify the nonzero-parameter cases. For $a < 0$, both modes decay, so the origin is a stable node and all initial states converge to it. For $a > 0$, the origin is a saddle; convergence occurs exactly when $m = 0$, that is, on $y = -x$. The $y = x$ mode then grows. There are no other equilibria in these two cases because $\det A_a = -2a \neq 0$.

Step 3: Resolve the equilibrium line. At $a = 0$, every point on $y = x$ is an equilibrium and $X_0(t) \rightarrow (m, m)^T$. This is the orthogonal projection of the initial state onto that line. The origin does not attract all nearby states: nearby nonzero points of the equilibrium line remain fixed. Nevertheless it is stable in the distance sense, since the orthogonal constant and decaying modes never increase the Euclidean norm.

Step 4: Compare the orders of limiting. For the specified data, $m = d = 1$. At each fixed $a < 0$ the time limit is zero; hence the first iterated limit is $\boxed{(0, 0)^T}$. At each fixed finite t , taking $a \uparrow 0$ instead gives $(1, 1)^T + e^{-2t}(1, -1)^T$, whose time limit is $\boxed{(1, 1)^T}$. The decay e^{at} becomes arbitrarily slow as a approaches zero from below. It is not uniform over such parameters on an unbounded time interval, so these two operations need not commute.