

Question 2

Consider

$$X' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} X.$$

Use the Euclidean distance to the line $y = -x$. In the perturbed initial data below assume $0 < \varepsilon < 1/\sqrt{2}$.

Tasks

1. Find the eigenpairs and general solution. Identify every initial state producing a bounded solution for $t \geq 0$.
2. Explain the saddle geometry and the forward directions on its two eigenlines. Can a nonconstant trajectory cross either eigenline?
3. For $X(0) = (1 + \varepsilon, -1 + \varepsilon)^T$, find when the magnitudes of its stable and unstable modal coefficients first become equal.
4. Find the first $t \geq 0$ at which this solution has distance 1 from $y = -x$. Explain why small initial error can eventually cause a large departure.

Question 2 – Solution

Strategy. Track stable and unstable coefficients separately; perpendicular distance measures departure from the stable line.

Step 1: Separate decay from growth. Eigenpairs are $3, (1, 1)^T$ and $-1, (1, -1)^T$. For data (p, q) ,

$$X = \frac{p+q}{2}e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{p-q}{2}e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

These independent vectors give the full family. A forward solution is bounded exactly when $p+q=0$, and then it tends to zero.

Step 2: Interpret the invariant lines. The unstable line $y=x$ points away from the origin; the stable line $y=-x$ points toward it. The opposite signs make the origin a saddle. Each modal coefficient preserves its sign unless identically zero. Thus off-line trajectories cannot cross either eigenline at finite time. Each equilibrium or nonzero half-line is a separate trajectory.

Step 3: Compute modal balance. The perturbed solution is $X = \varepsilon e^{3t}(1, 1)^T + e^{-t}(1, -1)^T$. Equality of modal magnitudes requires $\varepsilon e^{3t} = e^{-t}$, hence

$$t_{\text{balance}} = \frac{1}{4} \ln(1/\varepsilon).$$

The eigenvectors have equal length, so the same time balances their Euclidean contributions. It is nonnegative under the stated assumption.

Step 4: Measure the delayed departure. Distance to $y=-x$ is $|x+y|/\sqrt{2} = \sqrt{2}\varepsilon e^{3t}$, strictly increasing from a value below one. Its first value one occurs at

$$t_{\text{exit}} = \frac{1}{3} \ln\left(\frac{1}{\sqrt{2}\varepsilon}\right).$$

The delay diverges as $\varepsilon \downarrow 0$, but every positive error has a growing unstable component. Only exact cancellation stays on the stable line. The figure illustrates $\varepsilon = 1/10$.

