

Question 9

For $x' = 1 - x^2$, $y' = -y$, focus on the strip $-1 < x < 1$ and its boundaries. You may use $(\tanh t)' = 1 - \tanh^2 t$ and $\operatorname{artanh} x = \frac{1}{2} \ln((1+x)/(1-x))$ on this strip.

Tasks

1. Find the equilibria and verify the trajectory $(x, y) = (\tanh t, 0)$, including its direction and its limits as $t \rightarrow \pm\infty$.
2. Explain why the closed segment joining the two equilibria is not a single trajectory, even though it is the closure of the connecting orbit.
3. For initial state $(0, b)$, find a parametrization and eliminate time on the strip. Describe the forward and backward limits, treating $b = 0$ separately.
4. Determine trajectories on $x = -1$ and $x = 1$. Can a nonconstant one reach its equilibrium in finite time? Distinguish approaching an endpoint from attaining it.

Question 9 – Solution

Strategy. Keep equilibria separate from nearby nonconstant orbits, and retain the open domain when eliminating time.

Step 1: Identify the connecting trajectory. The equilibria are $(-1, 0), (1, 0)$. The function $(\tanh t, 0)$ solves both equations, has $x' > 0$, and tends to $(-1, 0)$ backward in time and $(1, 0)$ forward in time. Its orbit is the open horizontal segment $-1 < x < 1$, directed rightward.

Step 2: Separate the orbit from its closure. For every finite t , $-1 < \tanh t < 1$, so neither endpoint is reached. Each endpoint is its own constant trajectory. Uniqueness for the smooth field also prevents a solution from reaching an equilibrium and then leaving it. The closed segment is the union of three trajectories, not one.

Step 3: Recover the whole one-parameter family. For initial $(0, b)$,

$$x = \tanh t, \quad y = be^{-t}, \quad y = b\sqrt{\frac{1-x}{1+x}}, \quad -1 < x < 1.$$

The inverse hyperbolic tangent formula proves the eliminated equation. Every member tends to $(1, 0)$ as $t \rightarrow \infty$. If $b = 0$, the backward limit is $(-1, 0)$. If $b \neq 0$, then $x \rightarrow -1$ while $y \rightarrow +\infty$ for $b > 0$ or $-\infty$ for $b < 0$; there is no finite backward limiting state.

Step 4: Check the invariant boundary lines. On either $x = \pm 1$, the first coordinate stays fixed and $y = y_0 e^{-(t-t_0)}$. For $y_0 \neq 0$, each vertical half-line is an orbit directed toward $(\pm 1, 0)$, reached only as $t \rightarrow \infty$. At $y_0 = 0$ the state is already the equilibrium. The figure marks equilibria as separate filled points; connecting arrows and nearby trajectories approach them without asserting finite-time arrival.

$b = 1/2$ (solid), $b = -1/2$ (dashed), $b = 0$
Filled equilibria are not part of the connecting orbit.

