

Question 4

Compare the autonomous vector fields

$$F(x, y) = (-y, x), \quad G(x, y) = (1 + x^2 + y^2)(-y, x), \quad H(x, y) = (y, -x).$$

All three have the origin as an equilibrium. A geometric orbit and the time needed to traverse it are different pieces of information.

Tasks

1. Prove that every nonzero trajectory of each field lies on a circle centered at the origin.
2. On a circle of radius $R > 0$, compute the angular velocity for each field and determine its time direction.
3. Find the least positive period on that circle for each field. Give explicit solutions starting at $(2, 0)$ for F and G .
4. Explain precisely which information is shared by the three phase pictures and which requires arrows or time parametrization. Does a positive scalar multiple of a field reverse its directions?

Question 4 – Solution

Strategy. Compute radial and angular rates separately; multiplication of a field can change timing without changing its curves.

Step 1: Verify the common invariant circles. For F , $(x^2 + y^2)' = 2x(-y) + 2yx = 0$. Multiplication by $1 + x^2 + y^2$ preserves this cancellation for G , and changing the sign preserves it for H . A nonzero initial radius stays fixed and each field is nonzero on that circle.

Step 2: Compute orientation from angular velocity. For $R^2 = x^2 + y^2 > 0$,

$$\theta' = \frac{xy' - yx'}{R^2}.$$

Substitution gives $\theta'_F = 1$, $\theta'_G = 1 + R^2$, $\theta'_H = -1$. Thus F, G traverse circles counterclockwise, while H traverses them clockwise.

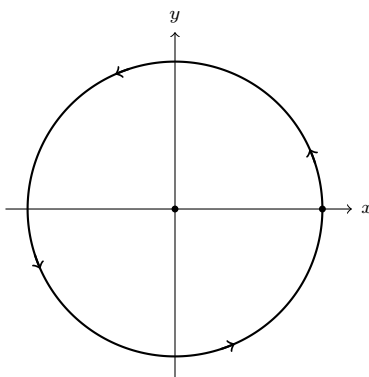
Step 3: Calculate periods and the given trajectories. A full revolution requires angular change of magnitude 2π , so

$$T_F = T_H = 2\pi, \quad T_G = \frac{2\pi}{1 + R^2}.$$

At radius 2, the requested solutions are $X_F = (2 \cos t, 2 \sin t)$ and $X_G = (2 \cos 5t, 2 \sin 5t)$. They start at the same state and cover the same circle at different rates. The zero equilibrium is not assigned a least positive revolution period.

Step 4: Distinguish geometry, direction and speed. All three fields have the same nonzero orbit sets and the same equilibrium. Arrows separate H from F, G ; the time parametrization separates F from G . A strictly positive multiplier preserves velocity direction wherever the field is nonzero, so it cannot reverse an orbit. A negative multiplier does reverse it. The plotted circle is shared by F, G ; its arrows show their common orientation, and the caption records their different periods at radius 2.

Same radius-2 orbit for F, G : counterclockwise



$T_F = 2\pi$, $T_G = 2\pi/5$; H reverses the arrows.