

Question 3

Consider the autonomous nonlinear system

$$x' = y - x^2, \quad y' = 1 - x.$$

A nullcline describes a zero coordinate velocity. It need not be a trajectory. No explicit solution of the full system is required.

Tasks

1. Find the equilibrium and both nullclines.
2. Use inequalities relative to the nullclines to determine the signs of x' , y' in every region. Give the actual velocity at $(0, 0)$, $(0, 1)$ and $(1, 0)$.
3. Determine tangent directions along each nullcline away from the equilibrium. Prove that neither whole nullcline is an invariant trajectory.
4. For a solution starting at $(0, 0)$, calculate $x'(0)$, $y'(0)$, $x''(0)$, $y''(0)$. Explain how it can have a vertical tangent while immediately leaving the x -nullcline.

Question 3 – Solution

Strategy. Read component signs from the two defining expressions, then distinguish instantaneous tangency from remaining on a curve.

Step 1: Locate the nullclines and their intersection. The x -nullcline is $y = x^2$, and the y -nullcline is $x = 1$. Their sole intersection is the equilibrium $(1, 1)$.

Step 2: Determine the regional directions. Above the parabola $x' > 0$; below it $x' < 0$. To the left of $x = 1$, $y' > 0$; to its right, $y' < 0$. Combining these independent signs gives right/up, left/up, right/down or left/down wherever the corresponding inequalities hold. The requested velocities are $F(0, 0) = (0, 1)$, $F(0, 1) = (1, 1)$ and $F(1, 0) = (-1, 0)$.

Step 3: Check nullcline tangency correctly. On $y = x^2$, away from $(1, 1)$, the nonzero velocity is vertical. The parabola has finite-slope tangent $(1, 2x)$, so this velocity is not tangent to that parabola. On $x = 1$, away from the equilibrium, the velocity is horizontal and not tangent to the vertical line. Thus neither complete nullcline is invariant. Each is crossed by typical solutions; the equilibrium is the exception where both velocities vanish.

Step 4: Examine departure from the origin. Differentiating the equations gives $x'' = y' - 2xx'$ and $y'' = -x'$. At the initial state,

$$x' = 0, \quad y' = 1, \quad x'' = 1, \quad y'' = 0.$$

Hence $x(t) = t^2/2 + o(t^2)$ and $y(t) = t + o(t^2)$ as $t \rightarrow 0$. For small positive t , $y - x^2 = t + o(t) > 0$, so the solution leaves the nullcline and acquires positive horizontal velocity. A vertical tangent at one instant does not mean that x stays constant. The figure shows nullclines and local velocity arrows, not guessed solution curves.

Solid: $y = x^2$
Dashed: $x = 1$; small arrows: velocities

