

Question 2

For $x' = x$, $y' = -y$, a student divides the equations to obtain $dy/dx = -y/x$ and concludes that every trajectory is a hyperbola $xy = C$. Investigate both the usefulness and the limitations of this calculation.

Tasks

1. Derive the conserved quantity without dividing by x or y . Find all equilibria.
2. Classify the trajectories lying on the coordinate axes, including their time directions. Explain what division can omit.
3. For $xy \neq 0$, identify the separate trajectory branches and their directions in all four quadrants. Explain why an entire two-branch hyperbola is not one trajectory.
4. Solve the IVP $(x(0), y(0)) = (2, 1/2)$ and describe both time limits. Determine all initial states whose solutions tend to the origin as $t \rightarrow \infty$.

Question 2 – Solution

Strategy. Keep zero-coordinate cases before separating variables, and use explicit signs to orient each connected branch.

Step 1: Find a division-free invariant. The product rule gives $(xy)' = x'y + xy' = xy - xy = 0$. Thus $xy = C$ along every solution, including axis solutions. The only equilibrium is $(0, 0)$.

Step 2: Recover the axis trajectories. If $y = 0$, then $x = x_0 e^t$; each nonzero horizontal half-axis is a trajectory directed away from the origin. If $x = 0$, then $y = y_0 e^{-t}$; each nonzero vertical half-axis is directed toward the origin. The origin itself is a constant trajectory. Division by x loses the vertical axis; subsequent division by y can also lose the horizontal axis.

Step 3: Separate the hyperbola branches. For nonzero coordinates, $x = x_0 e^t$ and $y = y_0 e^{-t}$ preserve both signs. In quadrants I, II, III, IV the directions are respectively right/down, left/down, left/up, right/up. Each fixed nonzero C has two disconnected branches; no continuous solution switches between them. The equation $xy = 0$ likewise combines several distinct trajectories rather than describing one orbit through the origin.

Step 4: Apply the initial data and characterize attraction. For the stated IVP,

$$x = 2e^t, \quad y = \frac{1}{2}e^{-t}, \quad xy = 1.$$

As $t \rightarrow \infty$, $x \rightarrow \infty$, $y \rightarrow 0$; as $t \rightarrow -\infty$, $x \rightarrow 0$, $y \rightarrow \infty$. Neither limit is the origin. For general initial data, convergence to the origin in forward time occurs exactly when $x_0 = 0$, including the zero state. The figure distinguishes the axis directions from representative hyperbola branches.

$xy = \pm 1$ and the four axis half-orbits

