

Question 1

Consider $x' = y$, $y' = -4x$, with initial state $(x, y) = (1, 0)$ at $t = 0$. A phase trajectory is the curve traced by $(x(t), y(t))$, with its direction as time increases. Nullclines are sets where one coordinate derivative is zero.

Tasks

1. Find the equilibrium and both nullclines. Determine the directions of the vector field on each coordinate axis away from the origin.
2. Find a conserved quadratic expression and the phase curve through the initial state. Explain why the curve must be drawn with equal coordinate scales.
3. Find a parametrization solving the IVP, its least positive period, and the direction in which the phase curve is traversed.
4. Locate every horizontal and vertical tangent on this phase curve. Explain why writing it globally as a single-valued function $y(x)$ would lose information.

Question 1 – Solution

Strategy. Use a conserved quantity to identify the geometric curve, then use the original equations to recover direction and timing.

Step 1: Locate zero derivatives. The x -nullcline is $y = 0$, and the y -nullcline is $x = 0$. Their only common point, $(0, 0)$, is the equilibrium. At $(x, 0)$ the field is $(0, -4x)$: downward for $x > 0$, upward for $x < 0$. At $(0, y)$ it is $(y, 0)$: rightward for $y > 0$, leftward for $y < 0$.

Step 2: Determine the conserved curve. Differentiating gives $(4x^2 + y^2)' = 8xy - 8xy = 0$. The initial state therefore lies on

$$4x^2 + y^2 = 4,$$

an ellipse with horizontal semiaxis 1 and vertical semiaxis 2. Equal coordinate scales preserve this actual geometric aspect ratio.

Step 3: Restore orientation and time. Direct substitution verifies $x = \cos(2t)$, $y = -2\sin(2t)$ and the initial data. The least positive period is π . At the initial point $(1, 0)$ the velocity is $(0, -4)$, so the ellipse is traversed clockwise. The conserved equation alone does not specify this orientation or period.

Step 4: Identify all tangent directions. Horizontal tangents occur where $y' = 0$, namely $(0, 2)$ and $(0, -2)$, where $x' \neq 0$. Vertical tangents occur where $x' = 0$, namely $(1, 0)$ and $(-1, 0)$, where $y' \neq 0$. For $-1 < x < 1$, the same x has two possible values $y = \pm 2\sqrt{1 - x^2}$ on this orbit. A single branch cannot represent the full closed trajectory. The origin is a separate equilibrium, not part of this ellipse. Arrows in the figure indicate increasing time.

