

Question 9

For

$$x' = y, \quad y' = -x,$$

only the measurements $x(0) = 1$ and $x(T) = b$ are given, where $T > 0$. The second initial component is unknown. You may verify solutions using sine and cosine identities; no phase-plane analysis is required.

Tasks

1. Find every solution consistent with $x(0) = 1$, using one arbitrary constant, and identify what that constant means.
2. Classify existence and uniqueness for all $T > 0$ and $b \in \mathbb{R}$, including the exceptional times $T = k\pi$.
3. Apply the classification to $(T, b) = (\pi, -1)$ and $(\pi/2, 2)$. Give the full solution set in each case.
4. When $\sin T \neq 0$, suppose the reported value of b has an error δ . Find the exact resulting error in the recovered second initial component. Explain what happens as $T \rightarrow \pi$ and why this is not failure of uniqueness for a fully specified IVP.

Question 9 – Solution

Strategy. Treat the unknown initial velocity as a parameter and ask whether the observations actually determine it.

Step 1: Parametrize all compatible initial states. Since $x'' = -x$ and $x(0) = 1$, the complete family is

$$x = \cos t + c \sin t, \quad y = -\sin t + c \cos t.$$

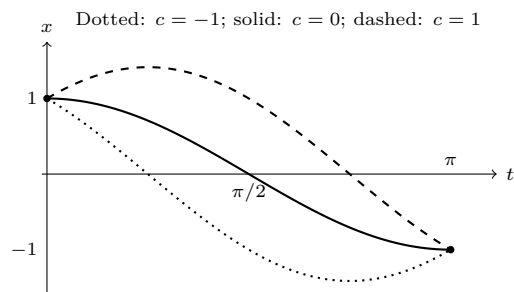
Direct differentiation verifies both rows. The arbitrary constant is $c = y(0) = x'(0)$, the missing initial component.

Step 2: Test the observation equation. The second measurement requires $\cos T + c \sin T = b$. If $\sin T \neq 0$, there is exactly one solution, with $c = (b - \cos T)/\sin T$. If $T = k\pi$ for a positive integer k , the measurement is independent of c : there are infinitely many solutions when $b = (-1)^k$, and none when $b \neq (-1)^k$. These cases exhaust all positive times.

Step 3: Evaluate the two specified experiments. At $(\pi, -1)$ every real c is allowed, giving the full family in Step 1. The figure shows three of these indistinguishable endpoint measurements. At $(\pi/2, 2)$, $c = 2$, so the unique solution is

$$X = (\cos t + 2 \sin t, -\sin t + 2 \cos t)^T.$$

Step 4: Quantify sensitivity and interpret it. Replacing b by $b + \delta$ changes c by exactly $\Delta c = \delta / \sin T$. Thus the absolute amplification factor is $1/|\sin T|$, which is unbounded as $T \rightarrow \pi$ through nonexceptional times. A fixed nonzero measurement error can then create a large recovered initial-velocity error. At $T = \pi$, that component is unobservable from these two measurements. Each fully specified initial vector still has a unique solution; the difficulty lies in recovering it from partial observations.



All share $x(0) = 1$ and $x(\pi) = -1$; initial velocities differ.