

Question 8

Let $R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $e_1 = (1, 0)^T$. Consider the periodically forced system

$$X' = AX + R(t)e_1.$$

You may use $R' = AR$, $R^{-1} = R(-t)$ and $R(t+2\pi) = R(t)$. A 2π -periodic solution satisfies $X(t+2\pi) = X(t)$ for all real t .

Tasks

1. Verify that $X_c(t) = R(t)(c + te_1)$ is a solution for every constant vector c . Determine its initial state.
2. Prove that this family contains every solution, by differentiating $R(-t)X(t)$.
3. Compute the change in state over one forcing period and decide whether any choice of initial data produces a 2π -periodic solution.
4. Replace e_1 in the forcing by an arbitrary constant vector $b = (\alpha, \beta)^T$. Find the full solution family and classify exactly when a 2π -periodic solution exists.

Question 8 – Solution

Strategy. Factor out the supplied rotating matrix and test periodicity on the full solution, not just on its coefficients and forcing.

Step 1: Verify the proposed family directly. The product rule gives

$$X'_c = AR(t)(c + te_1) + R(t)e_1 = AX_c + R(t)e_1.$$

Also $X_c(0) = c$. Thus every initial state has a member of this family.

Step 2: Prove completeness independently. Set $Z = R(-t)X$. Differentiating the inverse identity gives $(R^{-1})' = -R^{-1}A$. Consequently

$$Z' = -R^{-1}AX + R^{-1}(AX + Re_1) = e_1.$$

It follows that $Z = c + te_1$, so $X = R(t)(c + te_1)$ is the entire family, valid on the real axis.

Step 3: Examine one full forcing period. Periodicity of R gives

$$X(t + 2\pi) - X(t) = 2\pi R(t)e_1.$$

This difference has length 2π and never vanishes, independently of c . In particular $X(2\pi) = X(0) + (2\pi, 0)^T$. No initial state gives a 2π -periodic solution. A periodic forcing can produce a response with an accumulating change rather than a periodic response.

Step 4: Classify the general constant-amplitude forcing. The same calculation gives

$$X = R(t)(c + tb), \quad X(t + 2\pi) - X(t) = 2\pi R(t)b.$$

Since $R(t)$ is invertible, a periodic solution exists exactly when $b = 0$, that is, $\alpha = \beta = 0$. In this case every member $X = R(t)c$ is 2π -periodic, including the zero solution. If $b \neq 0$, no choice of c can cancel the per-period difference. The classification concerns this specified forcing family; it does not claim that every nonzero periodic forcing precludes a periodic response.