

Question 7

For a real parameter a , solve the nonlinear IVP

$$x' = x^2, \quad y' = -y, \quad x(0) = a, \quad y(0) = 1.$$

A maximal interval means the largest connected open time interval containing 0 on which the classical solution exists. The vector field is smooth.

Tasks

1. Derive the solution for $a \neq 0$, and handle $a = 0$ without dividing by the zero solution.
2. Find the maximal interval for each sign of a . Explain why smoothness of the field guarantees local uniqueness but not necessarily existence for all real times.
3. At $a = 1$, the rational formula is finite at $t = 2$. Does that value belong to a continuation of the stated IVP through $t = 1$? Justify your answer.
4. Compare $a = 0$ with $a = \varepsilon > 0$. Prove a small-data bound on any fixed $[0, T]$ when $\varepsilon T \leq 1/2$, and evaluate the difference at $t = 1/\varepsilon - 1$ for $0 < \varepsilon < 1$. Explain why the two conclusions are consistent.

Question 7 – Solution

Strategy. Derive explicit solutions, then distinguish the algebraic formula's domain from the connected interval of an initial-value solution.

Step 1: Solve and verify, retaining the zero case. For $a \neq 0$, separation gives $1/x = 1/a - t$, hence

$$X_a(t) = \left(\frac{a}{1-at}, e^{-t} \right)^T.$$

The first derivative is $a^2/(1-at)^2 = x^2$, and the data match at 0. For $a = 0$, the solution is $X_0 = (0, e^{-t})^T$; local uniqueness of the smooth field excludes other departures from $x = 0$.

Step 2: Find the maximal connected intervals. The first component blows up at $1/a$ when $a \neq 0$. Thus

$$a > 0 : (-\infty, 1/a); \quad a = 0 : \mathbb{R}; \quad a < 0 : (1/a, \infty).$$

The exponential component introduces no further restriction. A finite-valued continuous extension across the pole is impossible. Smoothness is a local regularity condition and does not prevent unbounded growth in finite time.

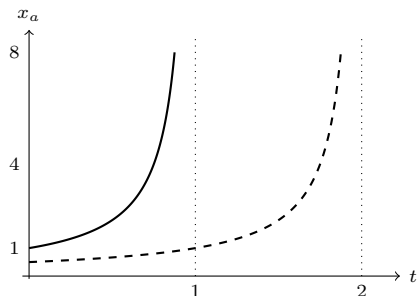
Step 3: Reject a disconnected continuation. At $a = 1$, the formula gives $x(2) = -1$, but the IVP through 0 exists only for $t < 1$. Its first component tends to $+\infty$ as $t \uparrow 1$. The branch for $t > 1$ solves the differential equation separately; it cannot be joined into a classical solution through the intervening pole.

Step 4: Compare finite-time and long-time sensitivity. For $0 \leq t \leq T$ with $\varepsilon T \leq 1/2$,

$$\|X_\varepsilon(t) - X_0(t)\| = \frac{\varepsilon}{1 - \varepsilon t} \leq 2\varepsilon.$$

But at $t = 1/\varepsilon - 1$, still before the pole, the difference equals 1. That comparison time tends to infinity as $\varepsilon \downarrow 0$. There is no contradiction: closeness on each fixed finite interval does not give a uniform bound over arbitrarily long, parameter-dependent times. The plot shows only the initial-value branches before their poles.

Solid: $a = 1$; dashed: $a = 1/2$
Dotted verticals: poles; $a = 0$ stays on the time axis.



Positive branches shown up to $x = 8$; no continuation across a pole.