

Question 5

Consider

$$X' = AX + f(t), \quad A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad f(t) = (1, 2e^{-t})^T.$$

The proposed particular solution is $P(t) = (1, 2e^{-t})^T$. Distinguish combinations of homogeneous solutions from combinations of solutions of this fixed nonhomogeneous system.

Tasks

1. Verify P , and prove that the full solution family is $P + c_1(e^{-t}, 0)^T + c_2(0, e^{-2t})^T$.
2. If U, V both solve the fixed forced system, determine exactly when $\alpha U + \beta V$ also solves it. Prove the condition for real constants α, β .
3. Solve the IVP $X(0) = (0, 0)^T$ and check its initial derivative against the original right-hand side.
4. For $t \geq 0$, find the maximum of the second component of this IVP and its time. Find the limiting state and decide whether that limiting vector, held constant, is itself a solution of the forced system.

Question 5 – Solution

Strategy. Subtract one particular solution to expose the homogeneous family; keep the forcing when testing superposition.

Step 1: Verify and complete the affine family. Here $P' = (0, -2e^{-t})^T = AP + f$. For any other solution X , the difference $H = X - P$ satisfies $H' = AH$, so $H = (c_1e^{-t}, c_2e^{-2t})^T$. Conversely each such sum solves the forced equation. Thus the proposed family is complete.

Step 2: Test a weighted combination. For $Z = \alpha U + \beta V$,

$$Z' - AZ = (\alpha + \beta)f(t).$$

Because the first component of f is always 1, equality to f holds exactly when $\boxed{\alpha + \beta = 1}$. In particular an average is a solution, while a sum generally is not. The difference of two forced solutions solves the homogeneous equation.

Step 3: Impose the zero initial state. The constants are $c_1 = -1$, $c_2 = -2$, yielding

$$\boxed{X(t) = (1 - e^{-t}, 2e^{-t} - 2e^{-2t})^T}.$$

At 0 this is zero, but $X'(0) = (1, 2)^T = f(0)$. A zero initial state therefore does not imply a zero solution in this nonhomogeneous problem.

Step 4: Locate the peak and interpret the limit. For $y = 2e^{-t}(1 - e^{-t})$, $y' = 2e^{-t}(2e^{-t} - 1)$. It is positive before $\ln 2$ and negative after, so the unique maximum on $t \geq 0$ is $\boxed{y(\ln 2) = 1/2}$. The state tends to $(1, 0)^T$. However the constant vector $(1, 0)^T$ has zero derivative while $A(1, 0)^T + f(t) = (0, 2e^{-t})^T \neq 0$ at finite times. A limiting state need not itself be a constant solution of a time-dependent forced equation. The plot marks the actual second-component peak.

