

Question 4

For the nonautonomous system

$$x' = 2ty, \quad y' = 0,$$

define the transition matrix $T(t, s)$ by $X(t) = T(t, s)X(s)$ for arbitrary real starting time s . It must satisfy $T(s, s) = I$.

Tasks

1. Derive $T(t, s)$ from the component equations and verify its differential equation and normalization.
2. Prove $T(t, r)T(r, s) = T(t, s)$ and determine the inverse of $T(t, s)$. Explain the role of the intermediate time.
3. Let $Y(t) = T(t, 0)$. Determine whether $Y(t + s) = Y(t)Y(s)$ holds for all real t, s , and give a concrete counterexample if it fails.
4. Solve the IVP $X(-1) = (2, 3)^T$. Compare $X(1)$ with $X(-1)$ and compute $X(0)$. Does either equality imply that the solution is constant?

Question 4 – Solution

Strategy. Keep both endpoint times: a time-dependent equation need not evolve according only to elapsed time.

Step 1: Integrate from the actual starting time. The second component is $y(t) = y(s)$, and $x(t) = x(s) + y(s) \int_s^t 2u \, du$. Hence

$$T(t, s) = \begin{pmatrix} 1 & t^2 - s^2 \\ 0 & 1 \end{pmatrix}.$$

Its derivative with respect to t is $\begin{pmatrix} 0 & 2t \\ 0 & 0 \end{pmatrix}$, which equals $A(t)T(t, s)$, and $T(s, s) = I$.

Step 2: Check composition and inversion. Multiplying two such shears adds their upper-right entries: $(t^2 - s^2) + (s^2 - r^2) = t^2 - r^2$. Thus

$$T(t, r)T(r, s) = T(t, s), \quad T(t, s)^{-1} = T(s, t).$$

The intermediate time cancels only because it is the end of the first journey and the start of the second, in the correct order.

Step 3: Test the one-parameter shortcut. The upper-right entry of $Y(t)Y(s)$ is $t^2 + s^2$, whereas that of $Y(t+s)$ is $(t+s)^2 = t^2 + s^2 + 2ts$. They agree only when $ts = 0$, not for all times. For example, $Y(1)Y(1)$ has upper-right entry 2, while $Y(2)$ has entry 4. This does not contradict the two-time composition identity.

Step 4: Apply the transition to the IVP. The solution is

$$X(t) = T(t, -1)(2, 3)^T = (3t^2 - 1, 3)^T.$$

It returns to $(2, 3)^T$ at $t = 1$, and $X'(0) = (0, 0)^T$. Nevertheless $X' = (6t, 0)^T$ is nonzero away from 0, so the solution is not constant. In a nonautonomous system, a repeated state at different times or a zero derivative at one instant does not force stationary evolution.