

Question 3

Consider the continuous time-dependent linear system

$$X' = A(t)X, \quad A(t) = \begin{pmatrix} 2t & 1 \\ 0 & -t \end{pmatrix}.$$

For two solution columns $U = (u_1, u_2)^T$, $V = (v_1, v_2)^T$, write $W = u_1v_2 - u_2v_1$. Do not assume a determinant evolution formula without deriving it.

Tasks

1. Differentiate W and use the original system to derive its scalar differential equation.
2. Find $W(t)$ when the matrix of columns equals I at $t = 0$. Is the claim $W = e^{t^2}$ consistent with the system?
3. Construct the principal fundamental matrix at 0. A definite integral is an acceptable exact entry; verify the initial matrix and both column equations.
4. Find the solution with initial state $(0, 1)^T$. Explain why growth of the fundamental determinant for $t > 0$ does not imply growth of every solution component.

Question 3 – Solution

Strategy. Derive the determinant law by the product rule, and construct the columns by solving the triangular equations.

Step 1: Differentiate the determinant. Substitution gives

$$W' = (2tu_1 + u_2)v_2 + u_1(-tv_2) - (-tu_2)v_1 - u_2(2tv_1 + v_2) = tW.$$

The off-diagonal terms cancel. This is the trace law here, since $\text{tr } A = t$, obtained directly rather than assumed.

Step 2: Apply the determinant's initial value. At 0, $W(0) = 1$, and solving $W' = tW$ gives $W(t) = e^{t^2/2}$. The proposed e^{t^2} has derivative $2te^{t^2}$ and fails the required equation except at the isolated time 0; matching the initial value is insufficient.

Step 3: Construct and verify the matrix. Put $J(t) = \int_0^t e^{-3s^2/2} ds$. The second equation gives $y = c_2 e^{-t^2/2}$, and the first gives $(e^{-t^2} x)' = c_2 e^{-3t^2/2}$. Therefore

$$\Phi(t) = \begin{pmatrix} e^{t^2} & e^{t^2} J(t) \\ 0 & e^{-t^2/2} \end{pmatrix}.$$

Since $J(0) = 0$, $\Phi(0) = I$. The derivative of its upper-right entry is $2te^{t^2} J + e^{-t^2/2}$, precisely the required first-row expression. The remaining entries satisfy their rows directly. Its determinant agrees with Step 2 and never vanishes, so this is a principal fundamental matrix.

Step 4: Interpret the selected solution. The initial vector $(0, 1)^T$ selects the second column: $X = (e^{t^2} J(t), e^{-t^2/2})^T$. Its second component strictly decreases for $t > 0$, even while W grows. A determinant concerns a pair of independent columns, not the separate monotonicity of each component. The nonzero determinant guarantees a complete independent family throughout the real axis.