

Question 2

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$\Phi(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix}.$$

A principal fundamental matrix at 0 is a fundamental matrix Y satisfying $Y(0) = I$. For real a , also define $R_a = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$ and $\Psi_a = \Phi R_a$.

Tasks

1. Verify $\Phi' = A\Phi$ and calculate its determinant. Explain why its columns describe a complete homogeneous solution family.
2. Construct the principal fundamental matrix $Y(t)$ and verify $Y(0) = I$. Express its entries using hyperbolic functions or exponentials.
3. For arbitrary initial data $X(0) = (p, q)^T$, find the constants in $X = \Phi c$ and the same solution in the form $X = YX(0)$.
4. Classify when Ψ_a is a fundamental matrix. At $a = 1$ and $a = -1$, determine exactly which initial states can still be represented by $X = \Psi_a d$, even though every column remains a solution.

Question 2 – Solution

Strategy. Distinguish solution columns from a complete basis, and normalize by the matrix at the initial time.

Step 1: Verify the columns and independence. Differentiating each exponential column gives $\Phi' = A\Phi$. Its determinant is -2 , never zero. Every initial vector can therefore be represented by $\Phi(0)c$; uniqueness of the linear IVP proves that the resulting constant combination supplies every solution.

Step 2: Normalize at the initial time. Since $\Phi(0)^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$,

$$Y(t) = \Phi(t)\Phi(0)^{-1} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.$$

Here $\cosh t = (e^t + e^{-t})/2$, $\sinh t = (e^t - e^{-t})/2$. Thus $Y(0) = I$ and $Y' = AY$ by differentiation or constant right multiplication.

Step 3: Recover arbitrary data. Solving $\Phi(0)c = (p, q)^T$ gives

$$c_1 = (p + q)/2, \quad c_2 = (p - q)/2, \quad X = (p \cosh t + q \sinh t, \quad p \sinh t + q \cosh t)^T.$$

This equals $Y(t)(p, q)^T$. The constants in a nonnormalized basis are not generally the initial state coordinates themselves.

Step 4: Detect a collapsed family. Constant combinations remain solutions, but $\det \Psi_a = -2(1 - a^2)$, so the family is fundamental exactly when $a \neq \pm 1$. At $a = 1$, both columns are $U + V$, whose value at 0 is $(2, 0)^T$; exactly the initial states $(p, 0)^T$ are representable. At $a = -1$, the columns are $U - V$ and its negative, with initial value $(0, 2)^T$; exactly $(0, q)^T$ are representable. Here U, V denote the columns of Φ . Neither singular case supplies all initial states, despite having two displayed solution columns.