

Question 1

For $X' = AX$, with $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, consider

$$U(t) = e^t(1, 0)^T, \quad V(t) = e^t(t, 1)^T, \quad W(t) = e^t(1, t)^T.$$

A vector solution must satisfy both rows at every time in its interval. A fundamental matrix has solution columns that form a basis at each time.

Tasks

1. Compute the residual $F' - AF$ for each candidate $F = U, V, W$. Decide which are solutions on a nonempty open interval.
2. Use the valid candidates to form a fundamental matrix. Prove that every solution is their constant linear combination, rather than merely checking that combinations work.
3. Solve the IVP $X(0) = (2, -1)^T$ and verify both the equations and initial data.
4. For this IVP, find the first zero of its first component for $t \geq 0$. Determine the full state and derivative there, and explain why a zero component does not mean a zero solution.

Question 1 – Solution

Strategy. Check vector residuals first; then solve the triangular equations to establish completeness.

Step 1: Test all candidates. Direct differentiation gives $U' - AU = 0$ and $V' - AV = 0$. For the third candidate,

$$W' = e^t(1, t+1)^T, \quad AW = e^t(1+t, t)^T, \quad \boxed{W' - AW = e^t(-t, 1)^T}.$$

Its second residual component never vanishes, so W is not a solution on any nonempty open interval.

Step 2: Prove that the valid family is complete. The matrix $\Phi = (U \ V) = e^t \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ has determinant $e^{2t} > 0$. More explicitly, the second equation gives $y = c_2 e^t$. Then $(e^{-t}x)' = c_2$, so

$$\boxed{X = e^t(c_1 + c_2 t, c_2)^T = c_1 U + c_2 V}.$$

This derivation reaches every solution and introduces exactly two arbitrary constants. The family is valid on the whole real axis.

Step 3: Select and verify the initial-value solution. At 0, the constants are $c_1 = 2$, $c_2 = -1$, hence $\boxed{X = e^t(2-t, -1)^T}$. Its derivative is $e^t(1-t, -1)^T$, exactly AX , and its initial state is $(2, -1)^T$ as required.

Step 4: Interpret a component zero. Since $e^t \neq 0$, the first component first vanishes at $t = 2$. There $X(2) = (0, -e^2)^T$ and $X'(2) = (-e^2, -e^2)^T$. The state is nonzero and continues moving; the first component crosses zero with negative slope. For a homogeneous linear system with continuous coefficients, only a full zero state at a time would force the identically zero solution by uniqueness. The plot displays both components.

