

Question 10

An unknown autonomous affine system has the form $X' = AX + b$ in $\mathbb{R}^2$, where A, b are constant. Independent measurements of its velocity field F give

$$F(0, 0) = (1, -1)^T, \quad F(1, 0) = (-1, 2)^T, \quad F(0, 1) = (2, -3)^T.$$

An affine system is written as a nonhomogeneous linear differential system; its constant forcing need not vanish.

Tasks

1. Recover A and b , and prove uniqueness within the class of affine systems. Explain why the measured inputs are states rather than initial derivatives of a single trajectory.
2. Find every equilibrium and translate it to the origin. Write the differential equation in the translated state and explain what happens to the constant forcing.
3. A fourth measurement reports $F(1, 1) = (0, 1)^T$. Is it compatible with the first three under the affine assumption? Quantify the discrepancy.
4. Construct a smooth nonlinear vector field fitting all four measurements. Explain why the first three determine an affine model uniquely but do not determine an unrestricted vector field.

Question 10 – Solution

Strategy. Subtract the velocity at the origin to isolate the matrix columns, then test the model assumption with new data.

Step 1: Recover the affine coefficients. At the origin, $b = F(0, 0) = (1, -1)^T$. The differences $F(1, 0) - b = (-2, 3)^T$ and $F(0, 1) - b = (1, -2)^T$ are the first and second columns of A . Therefore

$$A = \begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix}, \quad b = (1, -1)^T.$$

These columns and b are forced, proving uniqueness within the affine class. The measurements prescribe the field at three different states; they are not three independently assignable derivatives at one initial state.

Step 2: Translate the unique equilibrium. The equation $AX_* + b = 0$ reads $-2x + y + 1 = 0$ and $3x - 2y - 1 = 0$. It gives $X_* = (1, 1)^T$; uniqueness also follows from $\det A = 1$. For $Z = X - X_*$, the translation is constant, and

$$Z' = A(Z + X_*) + b = AZ, \quad \boxed{Z' = AZ.}$$

The forcing vanishes because the translation is by an equilibrium, not because every translation removes every forcing term.

Step 3: Check the fourth report. The recovered model predicts $F(1, 1) = (0, 0)^T$, while the report is $(0, 1)^T$. The reported-minus-predicted discrepancy is $(0, 1)^T$. No affine system fits all four measurements, since the first three already fix its coefficients. This conclusion tests the affine model assumption.

Step 4: Exhibit a nonlinear alternative. Define

$$\boxed{\tilde{F}(x, y) = (-2x + y + 1, 3x - 2y - 1 + xy)^T.}$$

The extra term xy vanishes at the first three states and equals 1 at $(1, 1)$, so all four measurements are met. This polynomial field is smooth and nonlinear. In it, $(1, 1)$ is not an equilibrium. More generally, fields obtained by adding $(0, cxy)^T$ all fit the first three data for any real c . Finite measurements uniquely identify this affine class, but leave unrestricted nonlinear models undetermined.