

Question 9

Three nonnegative dimensionless amounts undergo internal transfers: amount moves from compartment 1 to 2 at rate xy , and from 2 to 3 at rate $2y$. There are no external gains or losses. Their initial amounts are $(x, y, z) = (1, 1, 0)$. The rate laws are valid for nonnegative states.

Tasks

1. Derive the vector differential equation and classify it as autonomous and linear or nonlinear. Explain why a state-dependent coefficient does not make this a linear system.
2. Find the total-amount balance. Show that nonnegative initial states remain nonnegative and use this to bound every component for the stated data.
3. For the stated initial data, prove $x(t) > 0$ at every finite forward time. Derive a second conserved expression involving $x + y$ and $\ln x$.
4. Use the two conserved expressions to reduce the dynamics to one scalar equation for x , with formulas recovering y, z . State the restrictions that make the reduction meaningful; no explicit solution is required.

Question 9 – Solution

Strategy. Each internal transfer leaves one compartment and enters another at the same rate; this exposes conserved quantities before any solution is sought.

Step 1: Translate the transfers. The balances are

$$\boxed{x' = -xy, \quad y' = xy - 2y, \quad z' = 2y, \quad (x, y, z)(0) = (1, 1, 0).}$$

The system is autonomous, but nonlinear because of xy . Writing $x' = -yx$ does not make $-y$ a prescribed coefficient: y is another unknown state coordinate, not a known function of time.

Step 2: Prove conservation and nonnegativity. Adding the rows gives $(x + y + z)' = 0$, hence $x + y + z = 2$. Along any solution,

$$x(t) = x(0)e^{-\int_0^t y(s) ds}, \quad y(t) = y(0)e^{\int_0^t (x(s)-2) ds}, \quad z(t) = z(0) + 2 \int_0^t y(s) ds.$$

These formulas prove nonnegativity for nonnegative data on every forward existence interval. Thus $0 \leq x, y, z \leq 2$. The polynomial field is locally Lipschitz; boundedness prevents finite-time escape, so the solution continues for all $t \geq 0$.

Step 3: Obtain a second conserved quantity. With $x(0) = 1$ and $0 \leq y \leq 2$, the exponential formula gives $x(t) \geq e^{-2t} > 0$ at finite times. Therefore $\ln x$ is legitimate, and

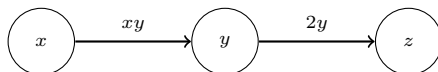
$$(x + y - 2 \ln x)' = -2y - 2(-xy)/x = 0.$$

Its initial value is 2, so $\boxed{x + y - 2 \ln x = 2}$.

Step 4: Reduce and recover the state. The two conservation laws give

$$\boxed{y = 2 - x + 2 \ln x, \quad z = -2 \ln x, \quad x' = -x(2 - x + 2 \ln x), \quad x(0) = 1.}$$

The physical branch has $x > 0$, $y \geq 0$, $z \geq 0$, in particular $x \leq 1$. Conversely, on this branch differentiation gives $y' = (x - 2)y$ and $z' = 2y$, proving recovery of the original system. Taking logarithms at $x = 0$ would be invalid; that boundary is excluded from this reduction.



Internal transfers only: $x + y + z = 2$ for the given data.