

Question 8

For $t \geq 0$, consider

$$x' = 2\sqrt{|x|}, \quad y' = -y, \quad x(0) = 0, \quad y(0) = 1.$$

You may use the local uniqueness theorem: continuity of a vector field and local Lipschitz continuity in the state guarantee local uniqueness. Failure of its hypotheses alone does not prove nonuniqueness.

Tasks

1. Check continuity and test the local Lipschitz condition at states with $x = 0$. Explain exactly which uniqueness guarantee is unavailable.
2. For any waiting time $a \geq 0$, construct a solution that has $x = 0$ until time a and then becomes positive. Verify the derivative at the joining time as well as both differential equations.
3. Find y and compare the initial state and initial derivative of the waiting-time solutions. Include the solution that never leaves $x = 0$.
4. If an additional observation says $x(1) = 0$, does that restore uniqueness for later times? Exhibit two distinct solutions satisfying all the data and explain what this example says about a complete initial state.

Question 8 – Solution

Strategy. Prove nonuniqueness by constructing different differentiable solutions; do not infer it solely from a theorem's failure.

Step 1: Test the regularity hypothesis. The field $F(x, y) = (2\sqrt{|x|}, -y)^T$ is continuous everywhere. For $h > 0$, the change in its first component between (h, y) and $(0, y)$, divided by the state distance, is $2\sqrt{h}/h = 2/\sqrt{h}$. This is unbounded as $h \downarrow 0$, so no local Lipschitz constant exists near $x = 0$. The quoted uniqueness theorem cannot guarantee uniqueness there.

Step 2: Construct and check the waiting family. For each $a \geq 0$, define

$$x_a(t) = \begin{cases} 0, & 0 \leq t \leq a, \\ (t-a)^2, & t > a. \end{cases}$$

Before a , both x'_a and $2\sqrt{|x_a|}$ are zero. After a , both are $2(t-a)$. At a , the left and right derivatives are zero, so the function is continuously differentiable and satisfies the equation there too. At $a = 0$, the same check uses the right derivative at the endpoint.

Step 3: Check the common data. The second scalar equation uniquely gives $y(t) = e^{-t}$. Every pair (x_a, e^{-t}) has initial state $(0, 1)$ and initial derivative $(0, -1)$. The pair $(0, e^{-t})$ is another solution, corresponding to waiting forever. Different finite waiting times produce different first components, despite identical initial states and slopes.

Step 4: Test whether one extra observation selects a solution. Every $a \geq 1$ gives $x_a(1) = 0$, as does the never-departing solution. For example, $a = 1$ and $a = 2$ satisfy all the data, but at $t = 3/2$ their first components are $1/4$ and 0 . Uniqueness is still absent. A complete state specifies all initial coordinates; unique evolution also requires appropriate properties of the differential equations. The figure shows three members of this family, with time increasing rightward.

