

Question 7

For a real constant κ , consider the implicit derivative equations

$$x' + y' = -x, \quad x' + \kappa y' = y.$$

A classical solution is a continuously differentiable pair satisfying both relations throughout an interval. An algebraically consistent derivative at one state is not necessarily part of a solution through that state.

Tasks

1. Write $M_\kappa X' = BX$ and determine exactly when it can be solved for a unique derivative at every state. Find the explicit system in that case.
2. At $\kappa = 1$, determine the states at which the two equations are algebraically consistent for a derivative. Describe the possible derivatives at such a state.
3. Still at $\kappa = 1$, find every classical solution on an interval of positive length. Explain why most instantaneously consistent states cannot be used as initial data.
4. For $\kappa \neq 1$ and initial state $(x, y) = (1, -1)$, compute the initial derivative and the derivative of $x + y$. Does a finite, parameter-independent initial velocity imply that a solution exists through the same state at $\kappa = 1$?

Question 7 – Solution

Strategy. Separate invertibility of the derivative matrix from the extra consistency conditions imposed when its rank drops.

Step 1: Solve when the derivative matrix is invertible. Here

$$M_\kappa = \begin{pmatrix} 1 & 1 \\ 1 & \kappa \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \det M_\kappa = \kappa - 1.$$

For $\kappa \neq 1$, inversion yields

$$X' = \frac{1}{\kappa - 1} \begin{pmatrix} -\kappa & -1 \\ 1 & 1 \end{pmatrix} X.$$

This constant linear system has a unique solution for every initial state.

Step 2: Test instantaneous algebraic consistency. At $\kappa = 1$, the left sides coincide, so the right sides must satisfy $y = -x$. At a state $(a, -a)$, the derivative pairs (p, q) satisfy $p + q = -a$, an entire line of possibilities. Outside $y = -x$, no derivative can satisfy both equations even at that instant.

Step 3: Enforce the constraint along a solution. A classical solution must have $y(t) = -x(t)$ at every time. Differentiating this identity gives $x' + y' = 0$. The first equation then forces $x = 0$, and the constraint forces $y = 0$. Therefore

$$X(t) \equiv (0, 0)^T \text{ is the only classical solution at } \kappa = 1.$$

It indeed satisfies both equations. Nonzero points on the consistency line admit instantaneous derivatives but no solution interval through them. Even at the origin, most algebraic derivative choices cannot persist.

Step 4: Check the limiting initial velocity. Substituting $(1, -1)$ into the explicit system gives $x' = (-\kappa + 1)/(\kappa - 1) = -1$ and $y' = 0$ for every $\kappa \neq 1$. Thus $(x + y)' = -1$, although initially $x + y = 0$. This velocity does not keep the state on the constraint line required at $\kappa = 1$. Its finiteness does not imply a limiting solution through $(1, -1)$; Step 3 proves that no such solution exists at that parameter.